THE PRICING OF INTEREST RATE CONTINGENT CLAIMS

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Abstract: The most important result in this working paper is the construction of a multi-dimensional Gaussian interest-rate term structure model, where, based on a construction of equivalent martingale measures and a suitable selection of numerators, it is shown that it is possible to derive analytical expressions for a wide range of derived instruments.

The instruments for which analytical expressions are derived in this connection are forward contracts, futures contracts, options on zero-coupon bonds, options on interest rates (including options on the slope and curvature of the term structure of interest rates), caps and floors, options on both forward contracts and futures contracts, options on CIBOR futures, including options on FRAs, the pricing of floating-rate bonds, swaps, swaptions and, finally, options on coupon bonds.

As regards the price expression for options on coupon bonds, a generalization is made of the Karoui, Myneni and Viswanathan (1993) model. The analytical expression derived here is namely a "real" analytical expression as opposed to Karoui, Myneni and Viswanathan which must be considered to be a semi-analytical expression

Using numerical tests we even managed to show that this new “true” closed form formula for the pricing of options on coupon bonds even seems to be valid in a general multi-factor Markovian HJM framework.

Keywords: Gaussian term-structure models, change of numeraire, closed-form solutions, HJM, Markovian structures, forward adjusted risk-measure

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The Pricing of Interest Rate Contingent Claims[1]

1. Introduction

The fundamental starting point for all pricing of interest-rate contingent claims is the term structure of interest rates. With regard to the pricing of the derived instruments, such as options on bonds, the existing literature has taken a number of different approaches, which may be said, generally, to belong to one of the following three categories.

The first category is what I would call local arbitrage-free term structure models, since these models are based on direct modelling of the bond prices themselves. This approach can therefore be considered an extension of the Black and Scholes (1973) model. Models that belong to this category include Ball and Torous (1983) and Schwartz and Schaefer (1987). An example of the problems that can arise in this context is given in Cheng (1991).

The second category is based on a description of the dynamics of the term structure of interest rates as a function of one or more state variables. However, these models do not only postulate that the dynamics of the term structure of interest rates can be defined by the state variables, but also that the initial term structure of interest rates is fully defined by them. Even though the models in this category can be further divided into models that set up a so-called equilibrium economy, such as Cox, Ingersoll and Ross (1985), Dothan (1978), Longstaff (1989) and Longstaff and Schwartz (1991) and models that postulate some sort of preference structure, such as Vasicek (1977), Brennan and Schwartz (1979) and Fong and Vasicek (1991), they can nonetheless all be contained in this category. In addition, for some of these models, it is impossible to deduce option price expressions that are independent of the preference structure which is not a desirable property see Heath, Jarrow and Morton (1991).

The third category, pioneered by Heath, Jarrow and Morton (1991), Jamshidian (1987) and Morton (1988) is based on the Martingale property in arbitrage-free markets. The idea underlying this modelling is first of all to define a term structure model, and next to choose a numerator, and finally to secure that under the change probability measure, each forward price will be an equivalent martingale along the lines of Harrison and Kreps (1979).

When considering the pricing of derived instruments, the natural approach is the third category, in that, firstly, these models take the term structure of interest rates for granted and then set up the appropriate arbitrage-free dynamics. Secondly, the numerator and the determination of equivalent martingale measures is the most straightforward approach, as will also appear from this working paper. Thirdly, and totally in line with Black and Scholes, once

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the volatitily structures are known, the models are fully described.

In this working paper, the pricing of derived instruments in a general term structure model in the Gaussian framework will be analyzed, the main objective being to make a generalisation of the Karoui, Myneni and Viswanathan (1993) option price model based on coupon bonds. With respect to this generalized formula we show by using numerical experiments that is even seems to be valid in a general multi-factor Markovian HJM framework.

The reason for focusing on the Gaussian framework is because of the possibility to obtain closed form solution for a wide range of contingent claims. In this connection, this paper is related to Babbs (1990), who also analyses the pricing of a wide range of derived instruments in the HJM framework.

I also intend to use the so-called Heath, Jarrow and Morton framework and then use a probability approach, together with a suitable selection of numerators, to derive analytical expressions for a wide range of different derived claims in a multi-factor framework. However, in this working paper, I intend to deal with European options only.

2. General properties of the term structure of interest rates

In this working paper I consider a continuous trading economy with zero-coupon bonds and a money market account with a trading interval $[0,\tau]$, for a fixed $\tau > 0$. In addition, it is assumed that money does not exist, i.e. that the agents in the economy are forced at all times to invest all their funds in assets. As usual, the uncertainty in the economy is characterized by the probability space (Ω,F,P), where Ω is the entire state space, P is a probability measure and F is the event space. At the same time, it is assumed that an m-dimensional Wiener process exists: $W = [W(t);0 < t \leq T < \tau]$, where the components $W_i(t)$, for $i = \{1,2,....,m\}$ are independent one-dimensional Wiener processes with a drift equal to zero (0) and a variance equal to one (1).

In addition, a continuum of zero-coupon bonds trade with different maturities T, for $T \in [0, \tau)$, where P(t,T) denotes the price at time t, for $t \in [0,T)$, of a zero-coupon bond expiring at time T.

In addition, it is a condition that $P(T,T) = 1$, which means that at the maturity date the bond must have a value equal to the face value.

The instanteneous forward rate at time t, for $T > t$, $r^F(t,T)$ is defined by:

$$r^F(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \qquad \textit{for all } T \in [0,\tau] \textit{ and all } t \in [0,T] \tag{1}$$

From this can be seen that $r^F(t,T)$ represents the rate that one can contract for at time t, for a risk-free investment in a forward contract that runs from time T to $T + \alpha$, for $\alpha \approx 0$.

The relation given by formula 1 means that the price of a zero-coupon bond P(t,T) can be written as:

$$P(t,T) = \exp\left(-\int_t^T r^F(t,s)ds \right) \qquad \textit{for all } T \; \varepsilon \; [0,\tau] \textit{ and all } t \; \varepsilon \; [0,T] \tag{2}$$

In addition, the spot rate at time t is given by the instantaneous forward rate of a forward contract that runs from t to t + α, i.e.:

$$r(t) = r^F(t,t) \qquad \textit{for all } t \; \varepsilon \; [0,\tau] \tag{3}$$

The bond price process also entails the presence of a yield curve at any time t, which can be written as:

$$\begin{gathered} R(t,T) = -\frac{\ln P(t,T)}{T - t} \qquad \textit{for } t < T \leq \tau \\ \textit{for} \\ P(t,T) = \exp[-R(t,T)(T - t)] \qquad \textit{for } t < T \leq \tau \end{gathered} \tag{4}$$

From this formula it can be seen that R(t,T) is the continuous yield to maturity of a zero-coupon bond at time t over the period [t,T].

The forward rate process at time T^F, observed at time t, for $t < T^F < T \leq \tau$, for a zero-coupon bond with maturity at time T means that[2]:

$$P(t,T^F,T) = P(T^F,T) = \frac{P(t,T)}{P(t,T^F)} \qquad \textit{for } t < T^F < T \leq \tau \tag{5}$$

In addition, this implies that the term structure of forward rates at time t across the interval $[T^F,T]$, is defined by:

$$R(t,T^F,T) = R(T^F,T) = -\frac{\ln P(T^F,T)}{T - T^F} \qquad \textit{for } t < T^F < T \leq \tau \tag{6}$$

where this means that the term structure of forward rates $r^F(t,T)$ is related to $R(T^F,T)$ in the following way:

$$r^F(t,T) = \lim_{(T^F \to T)} R(T^F,T) \qquad \textit{for } t < T^F < T \leq \tau \tag{7}$$

Finally, the price process for the money market account (i.e., the value of a unit that has a

[2] I will denote the price of a forward contract whenever appropriate as $P(t,T^F,T)$ or $P(T^F,T)$.

growth/capitalization factor that is given by the risk-free rate) is given by the following relation:

$$M(t) = e^{\int_0^t r(s)ds} \qquad for\ all\ t\ \varepsilon\ [0,\tau] \tag{8}$$

In this economy, the tradeable assets are given by the zero-coupon bonds, the money market account and the various derived instruments that can be constructed.

3: Definition of the yield-curve dynamic

The dynamic in the zero-coupon bond-prices P(t,T), for t < T ≤ τ, is assumed to be governed by an Ito process under the risk-neutral martingale measure Q:

$$\begin{gathered} \frac{dP(t,T)}{P(t,T)} = rdt - \sum_{i=1}^{m} \sigma_P(t,T;i)d\tilde{W}_i(t) \\ for \\ \tilde{W}_i(t) = W_i(t) - \Gamma_i(t) \end{gathered} \tag{9}$$

Where we have that P(0,T) is known for all T and P(T,T) = 1 for all T. Furthermore r is the risk-free interest rate, and $\sigma_P(t,T;i)$ represents the bond-price volatility, which can be associated with the i'th Wiener process, where $\tilde{W}_i$ is a Wiener process on (Ω,F,Q), for dQ = ρdP and ρ is the Radon-Nikodym derivative[3]. We also have that $\Gamma_i(t)$ represents the market-price of risk that can be associated with the i'th Wiener process.

In order to derive the following results it is not neccesary to assume that $\sigma_p(t,T;i)$ for i = {1,2,...m} is deterministic. It is sufficient to assume that $\sigma_p(t,T;i)$ is bounded, and its derivative (which is assumed to exist) is bounded.

Formula 9 can be rewritten as:

$$d\ln P(t,T) = \left[r - \frac{1}{2}\sum_{i=1}^{m} \sigma_P^2(t,T;i) \right] dt - \sum_{i=1}^{m} \sigma_P(t,T;i)d\tilde{W}_i(t) \tag{10}$$

The solution to this process can be expressed as:

[3] See Appendix A for a derivation of the arbitrage-free process.

$$\ln P(t,T) = \ln P(0,T) + \int_0^t \left[r(s) - \frac{1}{2}\sum_{i=1}^{m} \sigma_P^2(s,T;i) \right] ds - \sum_{i=1}^{m} \int_0^t \sigma_P(s,T;i)\, d\tilde{W}_i(s) \qquad \textbf{(11)}$$

and

$$0 = \ln P(0,t) + \int_0^t \left[r(s) - \frac{1}{2}\sum_{i=1}^{m} \sigma_P^2(s,t;i) \right] ds - \sum_{i=1}^{m} \int_0^t \sigma_P(s,t;i)\, d\tilde{W}_i(s) \qquad \textbf{(12)}$$

Where equation 12 follows from the horizon condition that P(T,T) = 1.

The drift in the process for the bond-price - r in formula 9 - can now be eliminated if we consider the difference between the process defined in formula 11 and the process that follows from the horizon condition (formula 12), ie:

$$\begin{aligned}\ln P(t,T) = \ln\frac{P(0,T)}{P(0,t)} &- \sum_{i=1}^{m}\int_0^t \frac{1}{2}[\sigma_P^2(s,T;i) - \sigma_P^2(s,t;i)]\,ds \\ &- \sum_{i=1}^{m}\int_0^t [\sigma_P(s,T;i) - \sigma_P(s,t;i)]\,d\tilde{W}_i(s)\end{aligned} \qquad \textbf{(13)}$$

An expression for the yield-curve R(t,T) can now be derived by using equation 4:

$$\begin{aligned}R(t,T) = R^F(0,t,T) &+ \sum_{i=1}^{m}\int_0^t \frac{1}{2}\left[\frac{\sigma_P^2(s,T;i) - \sigma_P^2(s,t;i)}{T - t}\right] ds \\ &+ \sum_{i=1}^{m}\int_0^t \left[\frac{\sigma_P(s,T;i) - \sigma_P(s,t;i)}{T - t}\right] d\tilde{W}_i(s)\end{aligned} \qquad \textbf{(14)}$$

The process for the forward-rates can also be derived - namely by using formula 2 and 13, ie:

$$r^F(t,T) = r^F(0,T) + \sum_{i=1}^{m}\int_0^t \sigma^F(s,T;i)\sigma_P(s,T;i)\,ds + \sum_{i=1}^{m}\int_0^t \sigma^F(s,T;i)\,d\tilde{W}_i(s) \qquad \textbf{(15)}$$

Where $\sigma^F(t,T;i)$ is defined as $\dfrac{\partial \sigma_P(t,T;i)}{\partial T}$, and can be recognized as being a measure for the forward rate volatility.

The spot-rate process is easily found from here:

$$r(t) \;=\; r^{F}(0,t) \;+\; \sum_{i=1}^{m} \int_{0}^{t} \sigma^{F}(s,t;i)\sigma_{P}(s,t;i)ds \;+\; \sum_{i=1}^{m} \int_{0}^{t} \sigma^{F}(s,t;i)d\tilde{W}_{i}(s) \tag{16}$$

That is, the spot-rate process is identical to the forward-rate process, except that in formula 16 we have a simultanous variation in the time-and maturity arguments.

As is obvious from equation 15 and 16, the stochastic processes for the interest rates are completely specified by the initial yield-curve and the volatility structure - which is exactly identical to the main result in the Heath, Jarrow and Mortons (1991) model framework.

This is in contrast to the traditional way the dynamic in the yield-curve has been modelled, see for example Cox, Ingersoll and Ross (1985), Vasicek (1977), Longstaff and Schwartz (1991), Beaglehole and Tenney (1991) and Langetieg (1980). In these models the process for one or more state-variables is specified and the yield-curve is now derived through the relationship between the spot-rate and the state-variables. The process for these state-variables fully specifies both the initial yield-curve and its dynamic. The initial yield-curve and its dynamic is now given when knowing the parameter vector that determines the process for the state-variables. This however, does not in general mean that the model yield-curve is equal to the actual yield-curve.

In practice there exist two different ways to ensure that these models are defined in accordance with the initial yield-curve. Firstly, we can determine the unknown parameter vector by using the so-called implied volatility approach from Brown and Schaefer (1994). Secondly, we can introduce a time-dependent parameter in the drift-specification for one of the state-variables - as in Hull and White (1990).

4. Pricing under the equivalent martingale measure

We have that the bond price P(t,T), i.e. the price at time t of a zero-coupon bond expiring at time T, can be written as follows under the equivalent martingale measure Q:

$$P(t,T) \;=\; E^{Q}\left[\exp\left(-\int_{t}^{T} r^{F}(t,s)ds\right)\right] \tag{17}$$

If we now consider a general bond, i.e. a coupon bond that is defined by a price process $P^k(t)$, and where the deterministic money flow is defined by a sequence of payments F_j at times T_j, for j = {1,2,....,n} and $t \leq T_1 < ..... < T_n < T$, then the price of this bond under the equivalent martingale measure is given by the following formula:

$$P^k(t) = \sum_{j=1}^{n} F_j P(t,T_j) \tag{18}$$

Let us then assume that there exists a European call option written on this coupon bond $P^k(t)$, with a strike price equal to X and an exercise date equal to T^F, for $T^F < T_1$, then this means that under the equivalent martingale measure the price of the option in question at time t can be written as follows:

$$C(t,T^F) = E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max[P^k(T^F) - X;0]\right] \tag{19}$$

Proposition no. 1

The price of this call option can be written as:

$$C(t,T^F) = \sum_{j=1}^{n} \Phi F_j P(t,T_j) Q_{T_j}(\Phi E) - \Phi X P(t,T^F) Q_{T^F}(\Phi E) \tag{20}$$

Where E = [$P(T^F,T) \geq X$], i.e. E is the event "the call is exercised", and Q_{Tj} is a probability measure, which is defined as below:

$$\frac{dQ_{T_j}}{dQ} = \frac{\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)}{E^Q\left[\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)\right]} \tag{21}$$

Where Q_{Tj} is an equivalent probability measure on Ω, which is equivalent to Q, so that the discounted value of any claim is a Q_{Tj}-martingale. In addition, it applies that Φ= 1 for call options and Φ= -1 for put options[4].

Proof:

Assuming $P(T_j,T)$, for $t < T_j < T$, is an expression of the standardized price of a zero- coupon bond $P(t,T)$, where there are no payments between the dates t and T_j and the standardization has been made with respect to the zero-coupon bond expiring at time T_j, i.e. $P(T_j,T)$ can be written as:

[4] This is based on the Gaussian assumption as it follows from the symmetrical property of the normal distribution.

$$P(T_j,T) = \frac{P(t,T)}{P(t,T_j)} \tag{22}$$

That $P(T_j,T)$ is a Q_{tj}-martingale can be seen in the following way:

From Appendix A we know that the price process for P(t,T) under the original probability measure Q, can be written as:

$$P(t,T) = E^{Q}\left[\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)P(T_j,T)\right] \tag{23}$$

By subsequently using the expression of the conditional expectations under the two probability measures Q and Q_{Tj}, from formula 21, we obtain:

$$E^{Q_{T_j}}[P(T_j,T)] = \frac{E^{Q}\left[P(T_j,T)\frac{dQ_{T_j}}{dQ}\right]}{E^{Q}\left[\frac{dQ_{T_j}}{dQ}\right]} \tag{24}$$

Thus, it is derived that the price under the probability measure Q_{tj} can be written as:

$$\begin{aligned} P(t,T) &= \left[\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)\right]E^{Q}\left[P(T_j,T)\frac{dQ_{T_j}}{dQ}\right] \\ &= \left[\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)\right]E^{Q}\left[\frac{dQ_{T_j}}{dQ}\right]E^{Q_{T_j}}[P(T_j,T)] \\ &= E^{Q_{T_j}}[P(T_j,T)]\left[\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)\right] \\ &\qquad \textit{or} \\ P(T_j,T) &= E^{Q_{T_j}}[P(T_j,T)] \end{aligned} \tag{25}$$

Where this completes the argument.

This argument deserves a comment. When we changed the probability measure from P to Q, the existing economy was replaced by an equivalent economy, where all market participants were risk-neutral. If no payments are made between t and T_j, the change of the probability measure from Q to Q_{Tj}, means that the term structure movements have been neutralized. Thus, if the prices have been standardized by applying formula 22, all prices will be martingales with respect to the new probability measure Q_{Tj}, and their standardized values will only be a *function of the volatility. Changing the probability measure from Q to Q_{Tj} therefore means that we have constructed an equivalent economy where market participants are time-neutral - i.e. the market participants have no preferences as concerns the present time.*

If I now rewrite the expression from formula 19 as:

$$C(t,T^F) = E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max[P^k(T^F) - X;0]\,|F_t\right] \tag{26}$$

then it is known from formula 18 that $P^k(T^F)$ can be expressed as follows:

$$P^k(T^F) = \sum_{j=1}^{n} F_j E^Q\left[\exp\left(-\int_{T^F}^{T_j} r^F(T^F,s)ds\right)|F_{T^F}\right] \tag{27}$$

By plugging formula 27 into formula 26 we obtain that $C(t,T^F)$ is given by the F_t expectation of:

$$\begin{aligned} C(t,T^F) &= 1_E\left[\sum_{j=1}^{n} F_j E^Q\left[\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)|F_{T^F}\right] - XE^Q\left[1_E\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)|F_{T^F}\right]\right] \\ &= \sum_{j=1}^{n} F_j E^Q\left[1_E\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)|F_t\right] - XE^Q\left[1_E\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)|F_t\right] \end{aligned} \tag{28}$$

where E is a F_{TF}-measurable event and 1_E represents its characteristic function[5].

By using the definition of $\frac{dQ_{T_j}}{dQ}$ *from formula 21, formula 28 can be rewritten as:*

$$C(t,T^F) = \sum_{j=1}^{n} F_j P(t,T_j) E^Q\left[\frac{dQ_{T_j}}{dQ}1_E|F_t\right] - XP(t,T^F)E^Q\left[\frac{dQ_{T^F}}{dQ}1_E|F_t\right] \tag{29}$$

formula 20 can now be derived[6]*, as* $E^Q\left[\frac{dQ_{T_j}}{dQ}1_E\right] = Q_{T_j}[E]$ *and* $\mathrm{E}^Q\left[\frac{dQ_{T^F}}{dQ}1_E\right] = Q_{T^F}[E]$,

which completes the argument.

[5] At this point it should be mentioned that the notation 1_E is to be attributed to the anticipated value of the approximation function with respect to the event E, i.e. $1_E(x) = 1$ for $x \varepsilon E$, and $1_E(x) = 0$ (zero) for $x \notin E$. In addition, see Lloyd (1980), section 12.4.

[6] See Karatzas and Shreve (1988) section 3.5.

Q.E.D.

However, this option formula is not particularly useful since additional specifications are required in order to permit calculation of the distribution of $P(T_j,T)$ under the probability measure Q_{Tj}. However, before rewriting formula 29, it is necessary to analyze this T_j - standardized price process in more detail.

5. The forward-adjusted pricing process

Proposition no. 2

Under the equivalent probability measure Q_{tj} the T_j-normalized pricing process $P^F(t,T_j,T)$, for $P^F(t,T_j,T) = \dfrac{P(t,T)}{P(t,T_j)}$, can be formulated as follows:

$$\frac{dP^F(t,T_j,T)}{P^F(t,T_j,T)} = \sum_{i=1}^{m} [\sigma_P(t,T;i) - \sigma_P(t,T_j;i)] d\tilde{W}_i^{Q_{T_j}}(t) \tag{30}$$

for $\tilde{W}_i^{Q_{T_j}}(t) = \tilde{W}_i(t) + \int_0^t \sigma_P(s,T_j;i) ds$

being a Q_{Tj} standardized Wiener process.

Proof:

It is known that P(t,T) is defined in the following way under the Q-standardized Wiener process:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\sum_{i=1}^{m} \int_0^t \sigma_P(s,T;i) d\tilde{W}_i(s) - \sum_{i=1}^{m} \int_t^T \int_0^t \sigma^F(s,v;i) \int_s^v \sigma^F(s,y;i) dy ds dv \right] \tag{31}$$

which can be rewritten as:

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left[-\sum_{i=1}^{m} \int_0^t \sigma_P(s,T;i) d\tilde{W}_i(s) - \sum_{i=1}^{m} \int_t^T \int_0^t \sigma^F(s,v;i) \sigma_P(s,v;i) ds dv \right] \\ &= \frac{P(0,T)}{P(0,t)} \exp\left[-\sum_{i=1}^{m} \int_0^t \sigma_P(s,T;i) d\tilde{W}_i(s) - \frac{1}{2} \sum_{i=1}^{m} \int_0^t \sigma_P^2(s,T;i) ds \right] \end{aligned} \tag{32}$$

This means that:

$$\frac{P(t,T)}{P(t,T_j)} = \frac{P(0,T)}{P(0,T_j)}\exp\left[-\sum_{i=1}^{m}\int_0^t[\sigma_P(s,T;i) - \sigma_P(s,T_j;i)]d\tilde{W}_i(s) - \frac{1}{2}\sum_{i=1}^{m}\int_0^t[\sigma_P^2(s,T;i) - \sigma_P^2(s,T_j;i)]ds\right] \qquad \textbf{(33)}$$

which by application of Ito´s lemma means that the process $dP^F(t,T_j,T)$ can be written as:

$$\frac{dP^F(t,T_j,T)}{P^F(t,T_j,T)} = \frac{1}{2}\sum_{i=1}^{m}[\sigma_P(t,T;i) - \sigma_P(t,T_j;i)]^2dt + \sum_{i=1}^{m}[\sigma_P(t,T;i) - \sigma_P(t,T_j;i)]d\tilde{W}_i(t) \qquad \textbf{(34)}$$

Now apply Girsanov's theorem in order to define a new probability measure.

$$\tilde{W}_i^{Q_{T_j}}(t) = \tilde{W}_i(t) + \int_0^t\sigma_P(s,T_j;i)ds,$$

which means that formula 34 can be written in the following form:

$$\frac{dP^F(t,T_j,T)}{P^F(t,T_j,T)} = \sum_{i=1}^{m}[\sigma_P(t,T;i) - \sigma_P(t,T_j;i)]d\tilde{W}_i^{Q_{T_j}}(t) \qquad \textbf{(35)}$$

Q.E.D.

Proposition no. 3

The bond price $P^F(t,T_j,T)$ under the probability measure Q_{Tj} can be expressed as:

$$P^F(t,T_j,T) = \frac{P(0,T)}{P(0,T_j)}\exp\left[-\sum_{i=1}^{m}\int_0^t[\sigma_p(s,T;i) - \sigma_P(s,T_j;i)]d\tilde{W}_i^{Q_{T_j}}(s) - \frac{1}{2}\sum_{i=1}^{m}\int_0^t[\sigma_P(s,T;i) - \sigma_P(s,T_j;i)]^2ds\right] \qquad \textbf{(36)}$$

for $\tilde{W}_i^{Q_{T_j}}(t) = \tilde{W}_i(t) + \int_0^t\sigma_P(s,T_j;i)ds.$

Proof:

According to Girsanov's theorem $\tilde{W}_i^{Q_{T_j}}(t)$ *is a Wiener process on (Ω,F,Q_{Tj}), where $dQ_{Tj} = \rho dQ$, and $P^F(t,T_j,T)$ satisfy the following stochastic integral:*

$$P^F(t,T_j,T) = \frac{P(0,T)}{P(0,T_j)}\exp\left[-\int_0^t\tilde{\mu}_P^{Q_{T_j}}(s,T_j,T)ds - \sum_{i=1}^{m}\int_0^t[\sigma_P(s,T;i) - \sigma_P(s,T_j;i)]d\tilde{W}_i^{Q_{T_j}}ds\right]$$

$$\textit{for} \qquad \textbf{(37)}$$

$$\tilde{\mu}_P^{Q_{T_j}}(t,T_j,T) = \sum_{i=1}^{m}\left[\frac{1}{2}[\sigma_P^2(t,T;i) - \sigma_P^2(t,T_j;i)] - \sigma_P(t,T_j;i)[\sigma_P(t,T;i) - \sigma_P(t,T_j;i)]\right]$$

Where $\sigma_P(t,T_j)$ is known to satisfy Novikov's theorem[7]*, which, in return, guarantees the existence of a probability measure Q_{Tj} which is equivalent to Q. That is* $\tilde{W}_i^{Q_{T_j}}(t) = \tilde{W}_i(t) + \int_0^t \sigma_P(s,T_j;i)ds$ *defines a* $\tilde{W}_i^{Q_{T_j}}(t)$ *- standardized Wiener process on* (Ω,F,Q_{Tj}).

In addition, it can be seen that the Q_{Tj}-risk neutral drift has the following form:

$$\tilde{\mu}_P^{Q_{T_j}}(t,T_j,T) = \frac{1}{2}\sum_{i=1}^{m}\left[\sigma_P(t,T;i) - \sigma_P(t,T_j;i)\right]^2$$

$$\Rightarrow \int_0^t \tilde{\mu}_P^{Q_{T_j}}(s,T_j,T)ds = \sum_{i=1}^{m}\left[\int_t^T\int_0^t \sigma^F(s,v;i)\sigma_P(s,v;i)dsdv + \int_t^{T_j}\int_0^t \sigma^F(s,v;i)\sigma_P(s,v;i)dsdv\right] - \sum_{i=1}^{m}\int_0^t\left[\int_s^T \sigma^F(s,v;i)dv\int_s^{T_j}\sigma^F(s,v;i)dv\right]ds \qquad \textbf{(38)}$$

Where plugging formula 38 into formula 37 yields formula 36, which completes the argument.

Q.E.D.

Now that the bond price process under the probability measure Q_{Tj} has been derived, it might also be interesting to define the spot rate and forward rate processes. However, the starting point here will be the spot rate process, since the forward rate process can be derived from this, as the argument below will show.

Proposition no. 4

Assuming that $r^F(t,T_j) = E^{Q_{T_j}}[r(T_j)]$, i.e. under the probability measure Q_{Tj} it applies that the spot forward rates are given by the expected spot rates at the time in question[8]**.**

Proof:

[7] This results from the definition of an equivalent martingale measure, where the requirement is that $\int_0^t \sigma_P(s,T)ds < \infty$, i.e. limited. As a matter of fact, this implicates that $E\left[e^{\frac{1}{2}\int_0^t \sigma_P^2(s,T)ds}\right] < \infty$, which is precisely identical to Novikov's theorem.

[8] This can be seen to be the local expectation theory, see Cox, Ingersoll and Ross (1981).

$$\begin{aligned} r^F(t,T_j) &= -\lim_{h \to 0} \frac{1}{h}\left[\frac{P(t,T_j+h) - P(t,T_j)}{P(t,T_j)}\right] \\ &= -\lim_{h \to 0} \frac{1}{h} E^{Q_{T_j}}[P(T_j,T_j+h) - 1] \\ &= E^{Q_{T_j}}[r(T_j)] \end{aligned} \tag{39}$$

Where the first line in the proof derives from rewriting and discretizing formula 1, the second line derives from using the T_j-bond as the numerator and line 3 follows directly.

Q.E.D.

Based on formulas 16 and 38, it can be seen that the spot rate process can be formulated as follows under the equivalent probability measure Q_{Tj}:

$$\begin{aligned} r(t) = r(0) &+ \sum_{i=1}^{m}\int_0^t \sigma^F(s,t;i)d\tilde{W}_i^{Q_{T_j}}(s) + \sum_{i=1}^{m}\int_0^t \sigma^F(s,t;i)\sigma_P(s,t;i)ds \\ &- \sum_{i=1}^{m}\int_0^t \sigma^F(s,t;i)\sigma_P(s,T_j;i)ds \end{aligned} \tag{40}$$

which entails that $r(T_j)$ can be written as:

$$r(T_j) = r^F(0,T_j) + \sum_{i=1}^{m}\int_0^{T_j} \sigma^F(s,T_j;i)d\tilde{W}_i^{Q_{T_j}}(s) \tag{41}$$

By applying proposition no. 4 it can be deduced that, when the T_j-period bond is chosen as numerator, the expected value of the spot rate at time T_j will be identical to the spot forward rate, i.e. $r^F(t,T_j)$ can be formulated as follows:

$$r^F(t,T_j) = r^F(0,T_j) + \sum_{i=1}^{m}\int_0^{t} \sigma^F(s,T_j;i)d\tilde{W}_i^{Q_{T_j}}(s) \tag{42}$$

In addition, an expression of the value of the forward rate at time T observed at time T_j can be found by using the price expression under the probability measure Q_{Tj}, which yields the following result:

$$r^F(T_j,T) = [r^F(0,T) - r^F(0,T_j)] + \sum_{i=1}^{m} \int_0^{T_j} \sigma^F(s,T;i) d\tilde{W}_i^{Q_{T_j}}(s) + \sum_{i=1}^{m} \left[\int_0^{T_j} \sigma^F(s,T;i)\sigma_P(s,T;i)ds - \int_0^{T_j} \sigma^F(s,T;i)\sigma_P(s,T_j;i)ds \right] \quad \textbf{(43)}$$

In addition, the forward rate process $r^F(t,T)$ under the probability measure Q_{Tj}, based on formula 40, can be written in the following form:

$$r^F(t,T) = r^F(0,T) + \sum_{i=1}^{m} \int_0^{t} \sigma^F(s,T;i) d\tilde{W}_i^{Q_{T_j}}(s) + \sum_{i=1}^{m} \left[\int_0^{t} \sigma^F(s,T;i)\sigma_p(s,T;i)ds - \int_0^{t} \sigma^F(s,T;i)\sigma_P(s,T_j;i)ds \right] \quad \textbf{(44)}$$

Based on the price process obtained in formula 36[9] and formula 4, and by using the boundary condition that P(T,T) = 1, the process for the term structure of interest rates under the equivalent martingale measure Q_{Tj} can be written in the following form:

$$R^F(t,t,T) = R^F(0,t,T) + \sum_{i=1}^{m} \int_0^{t} \frac{\sigma_P(s,T;i) - \sigma_P(s,t;i)}{T-t} d\tilde{W}_i^{Q_{T_j}}(s) + \sum_{i=1}^{m} \left[\int_0^{t} \left[\frac{\int_s^{T} \sigma^F(s,v;i)\sigma_P(s,v;i)dv - \int_s^{t} \sigma^F(s,v;i)\sigma_P(s,v;i)dv}{T-t} \right] ds - \int_0^{t} \frac{\sigma_P(s,T_j;i)[\sigma_P(s,T;i) - \sigma_P(s,t;i)]}{T-t} ds \right] \quad \textbf{(45)}$$

Under Q_{Tj}, defining $T = T_j$ in formula 45 means that the formula can be rewritten as follows:

9 However, rewritten to the process under the equivalent probability measure Q_{Tj} for $P^F(t,t,T)$, it results in the following formulation of $P^F(t,t,T)$:

$$P^F(t,t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\sum_{i=1}^{m} \int_0^{t} \sigma_P(s,T;i) d\tilde{W}_i^{Q_{T_j}}(s) + \sum_{i=1}^{m} \left[\int_0^{t} \sigma_P(s,T;i)\sigma_P(s,T_j;i)ds - \frac{1}{2}\int_0^{t} \sigma_P^2(s,T;i)ds \right] \right]$$

$$
\begin{aligned}
R^F(t,t,T_j) &= R^F(0,t,T_j) + \sum_{i=1}^{m} \int_0^t \frac{\sigma_P(s,T_j;i) - \sigma_P(s,t;i)}{T_j - t} d\tilde{W}_i^{Q_{T_j}}(s) \\
&+ \sum_{i=1}^{m} \left[\int_0^t \left[\frac{\int_s^{T_j} \sigma^F(s,v;i)\sigma_P(s,v;i)dv - \int_s^t \sigma^F(s,v;i)\sigma_P(s,v;i)dv}{T_j - t} \right] ds - \int_0^t \frac{\sigma_P(s,T_j;i)[\sigma_P(s,T_j;i) - \sigma_P(s,t;i)]}{T_j - t} ds \right] \\
&= R^F(0,t,T_j) + \sum_{i=1}^{m} \int_0^t \frac{\sigma_P(s,T_j;i) - \sigma_P(s,t;i)}{T - t} d\tilde{W}_i^{Q_{T_j}}(s) - \frac{1}{2}\sum_{i=1}^{m} \int_0^t \frac{(\sigma_P(s,T_j;i) - \sigma_P(s,t;i))^2}{T - t} ds
\end{aligned}
\tag{46}
$$

In addition, by defining $t = T_j$ in formula 45, $R^F(T_j,T_j,T)$, can be written as:

$$
\begin{aligned}
R^F(T_j,T_j,T) &= R^F(0,T_j,T) + \sum_{i=1}^{m} \int_0^{T_j} \frac{\sigma_P(s,T;i) - \sigma_P(s,T_j;i)}{T - T_j} d\tilde{W}_i^{Q_{T_j}}(s) \\
&+ \frac{1}{2}\sum_{i=1}^{m} \int_0^{T_j} \frac{(\sigma_P(s,T;i) - \sigma_P(s,T_j;i))^2}{T - T_j} ds
\end{aligned}
\tag{47}
$$

Thus, it can be derived that the expected value of $R^F(T_j,T_j,T)$ under Q_{Tj}, i.e. $E^{Q_{T_j}}[R^F(T_j,T_j,T)]$, is given by the initial forward rate structure $R^F(0,T_j,T)$ plus a risk premium of the form $\frac{1}{2}\sum_{i=1}^{m} \int_0^{T_j} \frac{(\sigma_P(s,T;i) - \sigma_P(t,T_j;i))^2}{T - T_j} ds$.

At this time, it might be appropriate to summarize some of the results. This I will do by showing the movements of the process for the bond prices, the forward rates and the spot rates under the two probability measures Q and Q_{Tj}.

Under the probability measure Q, these three processes can be written in the following form:

$$
\begin{aligned}
dP(t,T) &= r(t)P(t,T)dt + \sum_{i=1}^{m} \sigma_P(t,T;i)P(t,T)d\tilde{W}_i(t) \\
dr^F(t,T) &= \sum_{i=1}^{m} \sigma^F(t,T;i)\sigma_P(t,T;i)dt + \sum_{i=1}^{m} \sigma^F(t,T;i)d\tilde{W}_i(t) \\
dr(t) &= \sum_{i=1}^{m} \sigma^F(t,t;i)\sigma_P(t,t;i)dt + \sum_{i=1}^{m} \sigma^F(t,t;i)d\tilde{W}_i(t)
\end{aligned}
\tag{48}
$$

and under the probability Q_{Tj} they can be formulated as follows:

$$
\begin{aligned}
dP^F(t,t,T) &= [r(t) + \sum_{i=1}^{m} \sigma_P(t,T;i)\sigma_P(t,T_j;i)]P^F(t,t,T)dt + \sum_{i=1}^{m} \sigma_P(t,T;i)P^F(t,t,T)d\tilde{W}_i^{Q_{T_j}}(t) \\
dr^F(t,T) &= \sum_{i=1}^{m} \sigma^F(t,T;i)[\sigma_P(t,T;i) - \sigma_P(t,T_j;i)]dt + \sum_{i=1}^{m} \sigma^F(t,T;i)d\tilde{W}_i^{Q_{T_j}}(t) \\
dr(t) &= \sum_{i=1}^{m} \sigma^F(t,t;i)[\sigma_P(t,t;i) - \sigma_P(t,T_j;i)]dt + \sum_{i=1}^{m} \sigma^F(t,t;i)d\tilde{W}_i^{Q_{T_J}}(t)
\end{aligned}
\qquad \textbf{(49)}
$$

One last interesting observation should be made at this stage, namely that under the probability measure Q_{Tj}, $r^F(t,T)$ (and thus r(t)) is a martingale, which is not the case under the probability measure Q[10].

6. The pricing of interest-rate-contingent claims

Below I will define the price process for various derived claims; the instruments that will be analyzed are in section 5.1 forward and futures contracts on coupon bonds and forward and futures contracts on yields. In addition, it will be demonstrated how the listed CIBOR futures contract can be priced, and finally it will be demonstrated how FRAs can be priced in this framework.

Next, in section 5.2 I intend to derive a semi-analytical expression for options on coupon bonds, including the price expression for options written on zero-coupon bonds. In section 5.3, options written on yields will be considered, and price expressions for options written on the slope of the term structure and the curvature of the term structure.

Then in sections 5.4 and 5.5, I intend to analyze options on forward contracts and options on futures contracts, respectively. Directly after that I will in section 5.6 derive the price expression for options written on the listed CIBOR futures. In addition, the analytical expression of the price of an option written on FRAs will be analysed.

Section 5.7 will be devoted to the treatment of floating rate claims.

After that in section 5.8 I will do further analysis of the pricing of options on coupon bonds

The last three sections will be devoted to the pricing of swap derivatives. I will consider the following instruments: interest rate swaps, caps/floors and swaptions.

In the Gaussian framework some of these instruments has been considered in the litterature though in most cases the treatment has been less general than in our approach. Karoui, Myneni and Viswanathan (1993) consider the pricing of futures and forward contracts on both bonds

[10] This property is also attained under proposition no. 4, where precisely this property was used.

and yields and options on coupon bonds. Babbs (1990) treat futures and options on bonds and yields, FRNs, FRAs, swaps and swaptions. Jamshidian (1989) consider the pricing of options on coupon bonds in one-factor models - especially the Vasicek model. Hull and White (1994) analyse the pricing of Caps and Floors in the Hull and White model. In this connection it is worth emphasizing that the research which is closely related to ours is the work of Karoui, Myneni and Viswanathan.

It is of importance to point out that in all cases - except for the pricing equation for options on coupon bonds and swaptions - extension to an m-dimensional model is straightforward. Though, in section 7 using Monte Carlo simulation we get results which indicate that our new pricing equation for the valuation of options on coupon bonds even seems to be valid in a multi-factor Markovian setting.

6.1 Forward and futures contracts

It is assumed that there exists a forward contract on a coupon bond with settlement date at time T^F. This means that the price of the forward contract under the probability measure Q can be expressed as:

$$P^k(T^F) = \sum_{j=1}^{n} F_j E^Q\left[\exp\left(-\int_{T^F}^{T_j} r^F(T^F,s)ds\right)\right] \qquad for\ t < T^F < T_j \tag{50}$$

Under the probability measure Q_{TF}, this indicates that the forward price can be written in the following form[11]:

$$\begin{aligned}
P^k(T^F) &= \sum_{j=1}^{n} F_j E^{Q_{T^F}}\left[\exp\left(-\int_{T^F}^{T_j} r^F(T^F,s)ds\right)\right] \qquad for\ t < T^F < T_j \\
&= \sum_{j=1}^{n} F_j E^{Q_{T^F}}[P^F(T^F,T^F,T_j)] \\
&\qquad for \\
P^F(T^F,T^F,T_j) &= \frac{P(t,T_j)}{P(t,T^F)}\exp\left[-\frac{1}{2}\int_t^{T^F}[\sigma_P(s,T_j) - \sigma_P(s,T^F)]^2ds - \int_t^{T^F}[\sigma_P(s,T_j) - \sigma_P(s,T^F)]d\tilde{W}^{Q_{T^F}}(s)\right] \\
&\qquad and\ thus \\
E^{Q_{T^F}}[P^F(T^F,T^F,T_j)] &= \frac{P(t,T_j)}{P(t,T^F)} \qquad for\ j\ \varepsilon\ [1,2,....,n]
\end{aligned} \tag{51}$$

[11] In this derivation, the T^F -period bond has been used as numerator, and considering that I want to find the price of $P^F(T^F,T^F,T)$ at time T^F and not time t, integration has been made here over period t to T^F, and not period 0 to t. In addition, it is so that the price expression here has been formulated at time t and not as previously at time 0.

Where $P^F(T^F,T^F,T)$ is identical to $P^F(t,T^F,T)$ (from formula 36) observed[12] at time T^F.

Using the same principle, the price of a futures contract with settlement date at time T^F, on a coupon bond can be formulated as follows:

$$\begin{aligned}
P_F^k(T^F) &= \sum_{j=1}^{n} F_j E^Q\left[\exp\left(-\int_{T^F}^{T_j} r^F(T^F,s)ds\right)\right] \qquad for\ t < T^F < T_j \\
&= \sum_{j=1}^{n} F_j E^Q[P^F(T^F,T^F,T_j)] \\
&\qquad for \\
P^F(T^F,T^F,T_j) &= \frac{P(t,T_j)}{P(t,T^F)}\exp\left[-\frac{1}{2}\int_t^{T^F}[\sigma_P^2(s,T_j) - \sigma_P^2(s,T^F)]ds - \int_t^{T^F}[\sigma_P(s,T_j) - \sigma_P(s,T^F)]d\tilde{W}(s)\right] \\
&\qquad and\ thus \\
E^Q[P^F(T^F,T^F,T_j)] &= \frac{P(t,T_j)}{P(t,T^F)}\exp\left[\int_t^{T^F}[\sigma_P(s,T^F) - \sigma_P(s,T_j)]\sigma_P(s,T^F)ds\right] \qquad for\ j\ \varepsilon\ [1,2,....,n]
\end{aligned} \tag{52}$$

However, in contrast to what applied to the forward contract the price of the futures contract will be found under the probability measure Q instead of under Q_{TF}. The explanation is that under the probability measure Q_{TF} interest movements between time t and T^F have been eliminated. This would, of course, be inappropriate when futures contracts are analyzed, since they may be considered as a series of forward contracts that are continuously marked-to-market.

This observation enables us to specify the relationship between forward and futures contracts, as follows:

$$E^Q[P^F(T^F,T^F,T_j)] = \sum_{j=1}^{n} E^{Q_{T^F}}[P^F(T^F,T^F,T_j)]\exp\left[\int_t^{T^F}[\sigma_P^2(s,T^F) - \sigma_P(s,T_j)\sigma_P(s,T^F)]ds\right] \tag{53}$$

This relation shows precisely the difference between the expected values of the claims under the probability measure Q and the probability measure Q_{TF}, respectively.

Correspondingly, it is relatively elementary to find the price of forward and futures contracts written on the effective zero-coupon rate, which can be done by respectively rewriting the expression for $P^F(T^F,T^F,T)$ in formula 51 and 52 (for n=1) using formula 4.

If the CIBOR futures contract is to be priced in this framework, it can be done in the following

[12] A rather interesting observation can be made in this connection, namely that under the probability measure Q_{Tj}, the stochastic process $\dfrac{dP^F(T_v,T_j,T)}{P^F(T_v,T_j,T)}$ is identical for all T_v in the interval $t \le T_v \le T_j$, which implies that the price process $P^F(T_v,T_j,T)$ is also identical apart from the fact that the integration region follows T_v.

way. The price of the CIBOR future is defined as below:

$$P_C(T^F,T) = 1 - A(T^F,T) \qquad \textbf{(54)}$$

Where $A(T^F,T)$ represents the linear interest over the interval $[T^F,T]$[13], and $P_C(T^F,T)$ is the CIBOR futures price. In addition, $A(T^F,T)$ is defined thus:

$$A(T^F,T) = \left(\frac{1 + A(t,T)(T - t)}{1 + A(t,T^F)(T^F - t)} - 1\right)\frac{1}{T - T^F} \qquad \textbf{(55)}$$

Considering that there is the following relationship between A(t,T) and R(t,T); $A(t,T) = \dfrac{e^{R(t,T)(T - t)} - 1}{T - t}$, the price $P_C(T^F,T)$ can be determined to be formulated as follows under the probability measure Q:

$$\begin{aligned} P_C(T^F,T) &= 1 - E^Q[A^F(T^F,T^F,T)] \\ &\textit{for} \\ A^F(T^F,T^F,T) &= \left[\frac{1}{P^F(T^F,T^F,T)} - 1\right]\frac{1}{T - T^F} \\ &= \frac{1}{T - T^F}\left[\frac{P(t,T^F)}{P(t,T)}\exp\left[\frac{1}{2}\int_t^{T^F}[\sigma_P^2(s,T) - \sigma_P^2(s,T^F)]ds + \int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T^F)]d\tilde{W}(s)\right] - 1\right] \end{aligned} \qquad \textbf{(56)}$$

which results in $P_C(T^F,T)$ being formulated as follows:

$$P_C(T^F,T) = 1 - \frac{1}{T - T^F}\left[\frac{P(t,T^F)}{P(t,T)}\exp\left[\int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T^F)]\sigma_P(s,T^F)ds\right] - 1\right] \qquad \textbf{(57)}$$

which is the price of the listed CIBOR futures[14] contract for $T-T^F = 90/360$. T^F is the number of days until the second trading day before the third Wednesday of March, June, September and December, respectively (standardized in relation to 360) and assuming continuous marking-to-the-market. In addition, it applies that P(t,d), for $d = T - T^F$, is generally defined as: $P(t,d) = \dfrac{1}{1 + A(t,d)d}$.

6.2 Options on bonds

Let me just recapitulate the expression for the price of an option with an exercise date T^F written on a coupon bond, for $T^F < T_j$ and j = {1,2,.....,n}; hence:

---

[13] In practice, the interval $[T^F,T]$ is 3 months, i.e. 90/360.

[14] In the same way, it is possible to find the price of an FRA in this framework, since the FRA rate is given by the following relation: $FRA = E^{Q_{T^F}}[A^F(T^F,T^F,T)]$.

$$C(t,T^F) = \sum_{j=1}^{n} \Phi F_j P(t,T_j) Q_{T_j}(\Phi E) - \Phi X P(t,T^F) Q_{T^F}(\Phi E) \qquad \textbf{(58)}$$

where E = [$P(T^F,T) \geq X$], i.e. E is the event "the call is exercised", and Q_{Tj} is a probability measure which is defined as:

$$\frac{dQ_{T_j}}{dQ} = \frac{\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)}{E^Q\left[\exp\left(-\int_t^{T_j} r^F(t,s)ds\right)\right]} \qquad \textbf{(59)}$$

where Q_{Tj} is an equivalent probability measure on Ω, that is equivalent to Q, so that the discounted value of any claim is a Q_{Tj}-martingale.

However, this result can be changed and made more intuitive. As can be seen from formula 59, the event E = [$P(T^F,T) \geq X$], is defined both under the probability measure Q_{Tj} and the probability measure Q_{TF}, where it will be obvious to consider the option price under the same probability measure and where it will be natural to focus on the probability measure Q_{TF} - considering that time T^F is the exercise date of the option.

Proposition no. 5

The price of this option can be written in the following form:

$$C(t,T^F) = \sum_{j=1}^{n} \Phi F_j P(t,T_j) Q_{T^F}(\Phi E_j) - \Phi X P(t,T^F) Q_{T^F}(\Phi E) \qquad \textbf{(60)}$$

where:

$$E_j = \sum_{k=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - M_k + Cov(M_j,M_k)\right] \geq X$$
$$E = \sum_{k=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - M_k\right] \geq X$$
$$for$$
$$M_k = \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)] d\tilde{W}^{Q_{T^F}}(s) \tag{61}$$
$$V_k^2 = \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)]^2 ds$$
$$Cov(M_j,M_k) = \int_t^{T^F} [\sigma_P(s,T_j) - \sigma_P(s,T^F)][\sigma_P(s,T_k) - \sigma_P(s,T^F)] ds$$

Proof:

It is known from proposition no. 3 that, under the probability measure Q_{TF}, $P^F(T^F,T^F,T)$ observed at time T^F is defined as:

$$P^F(T^F,T^F,T) = \frac{P(t,T)}{P(t,T^F)} \exp\left[-\int_t^{T^F} [\sigma_p(s,T) - \sigma_P(s,T^F)] d\tilde{W}_i^{Q_{T^F}}(s) - \frac{1}{2}\int_t^{T^F} [\sigma_P(s,T) - \sigma_P(s,T^F)]^2 ds\right] \tag{62}$$

Alternatively, formula 62 can be written in the following form, as $\tilde{W}_i^{Q_{T^F}}$ is a Q_{TF} Wiener process:

$$P^F(T^F,T^F,T_k) = \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-M_k - \frac{1}{2}V_k^2\right] \tag{63}$$

Where it applies that M_k , for k = {1,2,....,n}, is a Gaussian vector under the probability measure Q_{TF}, and where M_k and V_k^2 are defined as in formula 61.

The event E is defined as:

$$\begin{aligned} E &= \left[\sum_{j=1}^{n} F_j P^F(T^F,T^F,T_k) \geq X\right]^Q = \left[\sum_{j=1}^{n} F_j P^F(T^F,T^F,T_k) \geq X\right]^{Q_{T^F}} \\ &= \left[\sum_{j=1}^{n} F_j \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-M_k - \frac{1}{2}V_k^2\right] \geq X\right]^{Q_{T^F}} \end{aligned} \tag{64}$$

In this connection, it is necessary to determine $P(T^F,T_j)Q_{T_J}(E)$, where:

$$P(t,T_j)Q_{T_j}(E) = E^{Q}\left[1_E \exp\left(-\int_t^{T_j} r^F(t,s)ds\right)\right] \quad \textbf{(65)}$$

As E ε F_{TF}, it is now possible to introduce the probability measure Q_{TF}, which means that formula 65 can be rewritten as:

$$\begin{aligned} P(t,T_j)Q_{T_j}(E) &= \left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\right] E^{Q_{T^F}}\left[1_E \exp\left(-\int_{T^F}^{T_j} r^F(T^F,s)ds\right)\right] \\ &= P(t,T^F)E^{Q_{T^F}}\left[1_E \frac{P(t,T_j)}{P(t,T^F)}\exp\left[-M_j - \frac{1}{2}V_j^2\right]\right] \\ \Rightarrow P(T^F,T_j)Q_{T_j}(E) &= \frac{P(t,T_j)}{P(t,T^F)}E^{Q_{T^F}}\left[1_E \exp\left[-M_j - \frac{1}{2}V_j^2\right]\right] \end{aligned} \quad \textbf{(66)}$$

The only unclear relation at this stage is the distribution of the vector M_k under the probability measure Q_{TF}.

Girsanov's theorem identifies the distribution under the new probability measure as:

$$\begin{aligned} P^F(T^F,T^F,T_k) = {} & \frac{P(t,T_k)}{P(t,T^F)}\exp\left[-\int_t^{T^F}[\sigma_p(s,T_k) - \sigma_P(s,T^F)]\left[d\tilde{W}_i^{Q_{T^F}}(s) + \int_t^{T^F}\sigma_P(s,T_j)ds - \int_t^{T^F}\sigma_P(s,T^F)ds\right]\right] \\ & \times \exp\left[-\left[\frac{1}{2}\int_t^{T^F}[\sigma_P(s,T_k) - \sigma_P(s,T^F)]^2 ds - \int_t^{T^F}\left[\sigma_P(s,T_k) - \sigma_P(s,T^F)\right]\left[\sigma_P(s,T_j) - \sigma_P(s,T^F)\right]ds\right]\right] \end{aligned} \quad \textbf{(67)}$$

which means that $Q_{Tj}(E)$ can be written as follows:

$$Q_{T_j}(E) = Q_{T^F}\left[\sum_{j=1}^{n} F_j \frac{P(t,T_k)}{P(t,T^F)}\exp\left[-M_k - \frac{1}{2}V_k^2 + Cov(M_j,M_k)\right]\right] \quad \textbf{(68)}$$

This implies that E_j can be expressed as:

$$E_j = \left[\sum_{j=1}^{n} F_j \frac{P(t,T_k)}{P(t,T^F)}\exp\left[-M_k - \frac{1}{2}V_k^2 + Cov(M_j,M_k)\right] \geq X\right]^{Q_{T^F}} \quad \textbf{(69)}$$

where this completes the argumentation.

Q.E.D.

Even though we have succeeded in principle in determing an analytical expression of the price of an option written on a coupon bond, one problem still remains, and that is to calculate the probabilities $Q_{TF}(E_j)$. In order to calculate these probabilities, it is necessary to calculate the distribution of a mix of log-normal variables, which is an intricate problem. However, there are cases where this calculation procedure is much simpler, and I intend to consider this case in section 6.8.

If a zero-coupon bond is being considered, i.e. n=1, this means that formula 60/61 can be reduced to:

$$
\begin{aligned}
C(t,T^F) &= \Phi P(t,T_k)Q_{T^F}(\Phi E_k) - \Phi XP(t,T^F)Q_{T^F}(\Phi E) \\
&\qquad \textit{for} \\
E_k &= \frac{P(t,T_k)}{P(t,T^F)}\exp\left[\frac{1}{2}V_k^2 - M_k\right] \geq X \\
E &= \frac{P(t,T_k)}{P(t,T^F)}\exp\left[-\frac{1}{2}V_k^2 - M_k\right] \geq X \\
&\qquad \textit{and} \\
M_k &= \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)]d\tilde{W}^{Q_{T^F}}(s) \\
V_k^2 &= \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)]^2 ds
\end{aligned}
\tag{70}
$$

Where this relation indicates the price of an option with the exercise date at time T^F written on a zero-coupon bond with maturity date at time T_k, for $t < T^F < T_k$.

As M_k, under the probability measure Q_{TF}, is a normally distributed variable with a mean value equal to 0 (zero) and a variance equal to V_K^2 the formula can be rewritten as follows, for $M_k \sim N(0,V_k^2)$, where N(*) is an expression of the cumulative normal distribution, hence:

$$
\begin{aligned}
E_k &= \left[M_k \leq \frac{1}{2}V_k^2 + \ln\frac{P(t,T_k)}{XP(t,T^F)}\right] \\
&\qquad \textit{and} \\
E &= \left[M_k \leq -\frac{1}{2}V_k^2 + \ln\frac{P(t,T_k)}{XP(t,T^F)}\right]
\end{aligned}
\tag{71}
$$

and finally:

$$C(t,T^F) = \Phi P(t,T_k)N(\Phi d_1) - \Phi XP(t,T^F)N(\Phi(d_1 - V_k))$$
$$for$$
$$d_1 = \frac{\ln\left(\dfrac{P(t,T_k)}{XP(t,T^F)}\right)}{V_k} + \frac{1}{2}V_k \tag{72}$$
$$and$$
$$V_k = \sqrt{\int_t^{T^F}[\sigma_P(s,T_k) - \sigma_P(s,T^F)]^2 ds]} = \sqrt{\int_t^{T^F}\left(\int_{T^F}^{T_k}\sigma^F(s,v)ds\right)^2 ds}$$

6.3 Options on interest rates

If, alternatively, we now consider price models for options on interest rates, a comparatively general formulation of the return profile could be expressed as follows:

$$C(t,T^F) = E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max[aR(T^F,T_x) + bR(T^F,T_y) + cR(T^F,T_z) - X;0]\right]$$
$$= P(t,T^F)E^{Q_{T^F}}\max[[aR(T^F,T_x) + bR(T^F,T_y) + cR(T^F,T_z) - X];0] \tag{73}$$
$$for\ t < T^F < T_x < T_y < T_z < T$$

For X being the exercise price (the exercise interest rate) and a, b, c ε R.

It appears for instance that for a = 1 and b,c = 0, this expression will degenerate to be an expression of the price of an option having the exercise date at time T^F on the zero-coupon yield for the period $[T^F,T_x]$. In addition, it can be seen that for a = -1, b = 1 and c = 0 formula 74 will define the price of an option on the slope of the term structure. For a call option, this mean a speculation in an increase in the slope of the term structure between $[T_x,T_y]$. In addition, it applies that for a,c = 1 and b = -1 formula 74 will be an expression of the price of an option on the curvature of the term structure (as so-called butterfly spread).

According to formula 47, $R(T^F,T)$ for $T = \{T_x,T_y,T_z\}$ under the probability measure Q_{TF}, can be written as:

$$R^F(T^F,T^F,T) = R^F(t,T^F,T) + \int_t^{T^F}\frac{\sigma_P(s,T) - \sigma_P(s,T^F)}{T - T^F}d\tilde{W}^{Q_{T_j}}(s)$$
$$+ \frac{1}{2}\int_t^{T^F}\frac{(\sigma_P(s,T) - \sigma_P(s,T^F))^2}{T - T^F}ds \tag{74}$$

here shown in the one-dimensional case.

If this option defined in formula 73 is now initially considered under the assumption that b = c = 0, this means that formula 73 can be written as:

$$\begin{aligned} C(t,T^F) &= E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max[aR(T^F,T_x) - X;0]\right] \\ &= P(t,T^F)E^{Q_{T^F}}\max[[aR(T^F,T_x) - X];0] \qquad for\ t < T^F < T_x < T \end{aligned} \tag{75}$$

Using the principles from propositions no. 1 and no. 5 it can then be shown that the option price can be formulated as follows:

$$\begin{gathered} C(t,T^F) = \Phi P(t,T^F)E^{Q_{T^F}}[aR(T^F,T)]Q_{T^F}(\Phi E_T) - \Phi XP(t,T^F)Q_{T^F}(\Phi E) \\ for \\ E_T = aR^F(t,T^F,T) - \frac{1}{2}V^2 + M \geq X \\ E = aR^F(t,T^F,T) + \frac{1}{2}V^2 + M \geq X \\ og \\ M = a\int_t^{T^F}\left[\frac{\sigma_P(s,T) - \sigma_P(s,T^F)}{T - T^F}\right]d\tilde{W}^{Q_{T^F}}(s) \\ V^2 = a^2\int_t^{T^F}\left[\frac{\sigma_P(s,T) - \sigma_P(s,T^F)}{T - T^F}\right]^2 ds \end{gathered} \tag{76}$$

As $M \sim N(0,V^2)$ and $aR(T^F,T) = -\dfrac{a}{T - T^F}\ln\left(\dfrac{P(t,T)}{P(t,T^F)}\right)$, this means that formula 76 can be rewritten as follows:

$$\begin{gathered} C(t,T^F) = \Phi aP(t,T^F)R^F(t,T^F,T)N(-d_1\Phi) - \Phi XP(t,T^F)N((-d_1 + V)\Phi) \\ for \\ d_1 = \frac{\dfrac{a}{T - T^F}\ln\left(\dfrac{P(t,T)}{P(t,T^F)P^X}\right)}{V} + \frac{1}{2}V \end{gathered} \tag{77}$$

Where V^2 is defined in formula 76 and P^X is the exercise rate converted to a price according to the following expression: $P^X = e^{-X(T - T^F)}$.

An interesting circumstance should be stressed here, namely that by comparing the call price of a zero-coupon yield (from formula 77) with the call price of a zero-coupon bond (from formula 72), it can be seen that the call price on a zero-coupon bond is equivalent to the put price on the corresponding zero-coupon yield[15] for a = 1 and $P(t,T) = aP(t,T^F)E^{Q_{T^F}}[R(T^F,T)]$. However, it is important to have in mind that the term $\frac{1}{2}V$ is to be standardized using (T - T^F), i.e. with the period from the exercise date of the option until the expiration day of the underlying instrument.

If we now consider the option defined in formula 73, then based on the principles highlighted in propositions no. 1 and no. 5 it can be shown, after lengthy calculations, that its analytical expression can be formulated as follows:

$$\begin{gathered}
C(t,T^F) = \sum_{L=1}^{3} \Phi P(t,T^F) E^{Q_{T_L}}[R(T^F,T_L)] Q_{T^F}(\Phi E_L) - \Phi X P(t,T^F) Q_{T^F}(\Phi E) \\
\mathit{for} \\
E = \sum_{k,u=1}^{3} v_k R^F(t,T^F,T_u) + M + \frac{1}{2}V^2 \geq X \\
E_L = \sum_{k,u=1}^{3} v_k R^F(t,T^F,T_u) + M + \frac{1}{2}V^2 - Cov_L \geq X \\
M = \sum_{k,u=1}^{3} v_k \int_t^{T^F} \frac{\sigma_P(s,T_u) - \sigma_P(s,T^F)}{T_u - T^F} d\tilde{W}_u^{Q_{T^F}} \\
V^2 = \sum_{k,u=1}^{3} v_k^2 \int_t^{T^F} \left(\frac{\sigma_P(s,T_u) - \sigma_P(s,T^F)}{T_u - T^F} \right)^2 ds \\
Cov_L = \sum_{k,u=1}^{3} v_k v_u \int_t^{T^F} \frac{[\sigma_P(s,T_u) - \sigma_P(s,T^F)][\sigma_P(s,T_L) - \sigma_P(s,T^F)]}{(T_u - T^F)(T_L - T^F)} ds \\
\mathit{and} \\
v_k \ \varepsilon \ [a,b,c] \ \mathit{and} \ u,L \ \varepsilon \ [x,y,z]
\end{gathered} \tag{78}$$

As:

[15] However, with the difference that the signs of P(t,T) and XP(t,T^F) are to be the same as if we were considering a call option.

$$
\begin{gathered}
E = [M \leq \frac{1}{V^2} + \sum_{k,u=1}^{3} v_k R^F(t,T^F,T_u) - X] \\
\Rightarrow Q_{T^F}(E) = N(d_2) \ ;as\ M \sim N(0,V^2) \\
for \\
d_2 = \frac{\sum_{k,u=1}^{3} v_k R^F(t,T^F,T_u) - X}{V} + \frac{1}{2}V \\
and \\
E_L = [M \leq \frac{1}{2}V^2 \sum_{k,u=1}^{3} v_k R^F(t,T^F,T_u) - X - Cov_L] \\
\Rightarrow Q_{T^F}(E_L) = N(d_1) \ ;as\ M \sim N(0,V^2) \\
for \\
d_1 = \frac{\sum_{k,u=1}^{3} v_k R^F(t,T^F,T_u) - X}{V} + \frac{1}{2}V - \frac{Cov_L}{V}
\end{gathered}
\qquad \textbf{(79)}
$$

This means that the option price can be written in the following form[16]:

$$
\begin{gathered}
C(t,T^F) = \sum_{L,k=1}^{3} \Phi P(t,T^F) v_k R^F(t,T^F,T_L) N(\Phi d_1) - \Phi X P(t,T^F) N\left(\Phi\left(d_1 + \frac{Cov_L}{V}\right)\right) \\
for \\
d_1 = \frac{\sum_{k,u=1}^{3} v_k R^F(t,T^F,T_u) - X}{V} + \frac{1}{2}V - \frac{Cov_L}{V}
\end{gathered}
\qquad \textbf{(80)}
$$

where, for v_k defined as in formula 78, and using suitable definitions of a,b and c, it can be seen to be an expression of the option price on the redemption yield, the slope of the term structure and the curvature of the term structure, respectively.

6.4 Options on forward contracts

Assuming that a call option has the exercise date at time T^F and this option has been written on a forward contract with settlement date at time T_1, where the underlying claim is a zero-coupon bond expiring at time T, for $t < T^F < T_1 < T$, the price of this option can be expressed as follows:

$$
C(t,T^F) = E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max\left[\frac{P(T^F,T)}{P(T^F,T_1)} - X;0\right]\right] \qquad \textbf{(81)}
$$

[16] Where in this derivation the convolution property for the normal distribution has been used.

Using proposition no. 1, the option price can be written in the following form:

$$C(t,T^F) = \Phi P(t,T^F) E^{Q_{T_1}}\left[\frac{P(T^F,T)}{P(T^F,T_1)}\right] Q_{T_1}(\Phi E) - \Phi P(t,T^F) X Q_{T^F}(\Phi E) \quad \textbf{(82)}$$

where, according to proposition no. 3, $E^{Q_{T_1}}\left[P^F(T^F,T_1,T)\right]$ is formulated as follows:

$$E^{Q_{T_1}}[P^F(T^F,T_1,T)] = \frac{P(t,T)}{P(t,T_1)} \quad \textbf{(83)}$$

The event E, for $E = \left[\frac{P(T^F,T)}{P(T^F,T_1)} \geq X\right]$ which is defined under the probability measure Q_{TF} can be written as:

$$E = \left[\frac{P(t,T)}{P(t,T_1)}\exp\left[-\frac{1}{2}\int_t^{T^F}[\sigma_P(s,T) + \sigma_P(s,T_1)]^2 ds + \int_t^{T^F}[\sigma_P(s,T^F) - \sigma_P(s,T_1)][\sigma_P(s,T) - \sigma_P(s,T_1)]ds - \int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T_1)]d\tilde{W}^{Q_{T^F}}\right] \geq X\right] \quad \textbf{(84)}$$

Using proposition no. 5, it can be seen that $Q_{T_1}(E) = Q_{T^F}(E_{T_1})$ for E_{T_1} defined thus:

$$E_{T_1} = \left[\frac{P(t,T)}{P(t,T_1)}\exp\left[-\frac{1}{2}\int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T_1)]^2 ds - \int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T_1)]d\tilde{W}^{Q_{T^F}}\right] \geq X\right] \quad \textbf{(85)}$$

Which indicates that the option price can be expressed in the following way:

$$C(t,T^F) = \Phi P(t,T^F) E^{Q_{T_1}}\left[\frac{P(T^F,T)}{P(T^F,T_1)}\right] Q_{T^F}(\Phi E_{T_1}) - \Phi P(t,T^F) X Q_{T^F}(\Phi E) \quad \textbf{(86)}$$

Consequently, the option price can finally be written as:

$$C(t,T^F) = \Phi P(t,T^F)\frac{P(t,T)}{P(t,T_1)}N(\Phi d_1) - \Phi P(t,T^F)XN(\Phi(d_1 + W))$$
$$for$$
$$d_1 = \frac{\ln\left(\frac{P(t,T)}{P(t,T_1)X}\right)}{V} - \frac{1}{2}V$$
$$V^2 = \int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T_1)]^2 ds \qquad (87)$$
$$W = \frac{\int_t^{T^F}[\sigma_P(s,T^F) - \sigma_P(s,T_1)][\sigma_P(s,T) - \sigma_P(s,T_1)]ds}{V}$$

6.5 Options on futures contracts

If, as opposed to the previous section, we now consider a call option with exercise date at time T^F which is written on a futures contract with settlement date at time T_1, where the underlying claim is a zero-coupon bond expiring at time T, for $t < T^F < T_1 < T$, then the price of this option can be expressed as:

$$C(t,T^F) = E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max\left[\frac{P(T^F,T)}{P(T^F,T_1)} - X;0\right]\right] \qquad (88)$$

Using proposition no. 1, this means that the option price can be written in the following form:

$$C(t,T^F) = \Phi P(t,T^F)E^Q\left[\frac{P(T^F,T)}{P(T^F,T_1)}\right]Q(\Phi E) - \Phi P(t,T^F)XQ_{T^F}(\Phi E) \qquad (89)$$

where, according to proposition no. 3, $E^Q\left[P^F(T^F,T_1,T)\right]$ can be formulated as follows:

$$E^Q[P^F(T^F,T_1,T)] = \frac{P(t,T)}{P(t,T_1)}\exp\left[\int_t^{T^F}[\sigma_P(s,T_1) - \sigma_P(s,T)]\sigma_P(s,T_1)ds\right] \qquad (90)$$

The event E, for $E = \left[\frac{P(T^F,T)}{P(T^F,T_1)} \geq X\right]$ which is defined under the probability measure Q_{TF}, can be written as:

$$E = \left[\frac{P(t,T)}{P(t,T_1)} \exp\left[-\frac{1}{2}\int_t^{T^F} [\sigma_P(s,T) + \sigma_P(s,T_1)]^2 ds + \int_t^{T^F} [\sigma_P(s,T^F) - \sigma_P(s,T_1)][\sigma_P(s,T) - \sigma_P(s,T_1)] ds - \int_t^{T^F} [\sigma_P(s,T) - \sigma_P(s,T_1)] d\tilde{W}^{Q_{T^F}} \right] \geq X \right] \qquad \textbf{(91)}$$

Using proposition no. 5, it can be seen that $Q(E) = Q_{T^F}(E_Q)$ for E_Q is defined thus:

$$E_Q = \left[\frac{P(t,T)}{P(t,T_1)} \exp\left[-\frac{1}{2}\int_t^{T^F} [\sigma_P(s,T) - \sigma_P(s,T_1)]^2 ds - \int_t^{T^F} \sigma_P(s,T_1)[\sigma_P(s,T) - \sigma_P(s,T_1)] ds - \int_t^{T^F} [\sigma_P(s,T) - \sigma_P(s,T_1)] d\tilde{W}^{Q_{T^F}} \right] \geq X \right] \qquad \textbf{(92)}$$

This indicates that the option price can be expressed in the following way:

$$C(t,T^F) = \Phi P(t,T^F) E^Q \left[\frac{P(T^F,T)}{P(T^F,T_1)} \right] Q_{T^F}(\Phi E_Q) - \Phi P(t,T^F) X Q_{T^F}(\Phi E) \qquad \textbf{(93)}$$

which, in conclusion, means that the option price can be written as:

$$\begin{gathered} C(t,T^F) = \Phi P(t,T^F) \frac{P(t,T)}{P(t,T_1)} \exp\left[\int_t^{T^F} [\sigma_P(s,T_1) - \sigma_P(s,T)] \sigma_P(s,T_1) ds \right] N(\Phi d_1) \\ - \Phi P(t,T^F) X N(\Phi(d_1 + W(T^F))) \\ \textit{for} \\ d_1 = \frac{\ln\left(\frac{P(t,T)}{P(t,T_1)X} \right)}{V} - \frac{1}{2}V - W(T_1) \\ V^2 = \int_t^{T^F} [\sigma_P(s,T) - \sigma_P(s,T_1)]^2 ds \\ W(\tau) = \frac{\int_t^{T^F} \sigma_P(s,\tau)[\sigma_P(s,T) - \sigma_P(s,T_1)] ds}{V} \end{gathered} \qquad \textbf{(94)}$$

An interesting point should be mentioned here, namely that the event E is identical for options on forward contracts and options on futures contracts. The explanation is that when considering the relative price process $\frac{P(T^F,T)}{P(T^F,T_1)}$, a change from the probability measure Q to Q_{TF} yields a result that is equivalent to the probability measure being changed from Q_{T1} to Q_{TF}.

6.6 Options on CIBOR futures

Pricing an option written on the CIBOR future that was considered in section 6.1 can be done as shown below; but let me first recapitulate the price expression in formula 57 for this CIBOR future; thus:

$$P_C(T_1,T) = 1 - \frac{1}{T-T_1}\left[\frac{P(t,T_1)}{P(t,T)}\exp\left[\int_t^{T_1}[\sigma_P(s,T) - \sigma_P(s,T_1)]\sigma_P(s,T_1)ds\right] - 1\right] \qquad \textbf{(95)}$$

This relation indicates the price at time t of a CIBOR future with the settlement date being time T_1, and the underlying instrument being the $[T_1 - T]$ rate.

Assuming now that an option has been written on this CIBOR future with exercise date at time T^F, for $t < T^F < T_1 < T$, then the price of this option can be written as:

$$\begin{aligned} C(t,T^F) &= \exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max\left[\left(1 - \frac{1}{T-T_1}\left(\frac{1 + A(T^F,T)(T - T^F)}{1 + A(T^F,T_1)(T_1 - T^F)} - 1\right)\right) - X;0\right] \\ &= E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\left(1 - \frac{1}{T-T_1}\left(\frac{1 + A(T^F,T)(T - T^F)}{1 + A(T^F,T_1)(T - T_1)} - 1\right)\right)\right] \\ &\quad - E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)X\right] \end{aligned} \qquad \textbf{(96)}$$

Using the principles in propositions no. 1 and no. 2, and the results derived in connection with options on futures contracts in section 6.5, it can be shown[17] that in this connection the option price can be written in the following form:

$$\begin{aligned} C(t,T^F) &= \Phi P(t,T^F)\left[1 - \frac{1}{T-T_1}\left[\left(\frac{1 + A(t,T)(T - t)}{1 + A(t,T_1)(T_1 - t)}\right)\exp\left(\int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T_1)]\sigma_P(s,T_1)ds\right) - 1\right]\right]N(\Phi d_1) \\ &\quad - \Phi P(t,T^F)XN(\Phi(d_1 + W(T^F))) \\ &\textit{for} \\ d_1 &= -\frac{\ln\left(\frac{1 + A(t,T)(T - t)}{(1 + (1 - X)(T - T_1))(1 + A(t,T_1)(T_1 - t))}\right)}{V} - \frac{1}{2}V - W(T_1) \\ V^2 &= \int_t^{T^F}[\sigma_P(s,T) - \sigma_P(s,T_1)]^2 ds \\ W(\tau) &= \frac{\int_t^{T^F}\sigma_P(s,\tau)[\sigma_P(s,T) - \sigma_P(s,T_1)]ds}{V} \end{aligned} \qquad \textbf{(97)}$$

where X (the exercise price) is defined as a CIBOR futures price according to the relation given by formula 54.

[17] Where this proof is left to the reader.

Correspondingly, it is possible to find an analytical expression of the price of an option written on an FRA, where it can be seen that this will be of the following form:

$$
\begin{aligned}
C(t,T^F) &= \Phi P(t,T^F)\left[\frac{1}{T - T_1}\left[\left(\frac{1 + A(t,T)(T - t)}{1 + A(t,T_1)(T_1 - t)}\right) - 1\right]\right]N(\Phi d_1) - \Phi P(t,T^F)XN(\Phi(d_1 - W)) \\
&\qquad\qquad \textit{for} \\
d_1 &= \frac{\ln\left(\dfrac{1 + A(t,T)(T - t)}{(1 + (1 - X)(T - T_1))(1 + A(t,T_1)(T_1 - t))}\right)}{V} + \frac{1}{2}V \\
V^2 &= \int_t^{T^F} [\sigma_P(s,T) - \sigma_P(s,T_1)]^2 ds \\
W &= \frac{\int_t^{T^F} [\sigma_P(s,T^F) - \sigma_P(s,T_1)][\sigma_P(s,T) - \sigma_P(s,T_1)]ds}{V}
\end{aligned}
\qquad \textbf{(98)}
$$

Formula 98 yields the price at time t of an option with exercise day at time T^F, and where the underlying claim is an FRA for the period $[T_1,T]$, for $t < T^F < T_1 < T$.

6.7 Floating-rate claims

In this framework floating-rate claims can also be priced as a sum of the prices of the individual floating-rate-contingent payments.

If, initially, we consider a floating-rate claim with a maturity date at time T_2, where the interest payment at this time is to be calculated over the period $[T_1,T_2]$, and where the interest rate that applies for the period in question is known at time T_0, then this means that the price of this cash flow (this claim) can be expressed as:

$$
\begin{aligned}
P_V(t,T_2) &= E^Q\left[\left(\exp-\int_t^{T_2} r^F(t,s)ds\right)[H + H(T_2 - T_1)R(T_0;T_1,T_2)]\right] \\
&= P(t,T_2)E^{Q_{T_2}}[H + H(T_2 - T_1)R(T_0;T_1,T_2)] \qquad \textit{for } t < T_0 < T_1 < T_2
\end{aligned}
\qquad \textbf{(99)}
$$

where H is the principal and $R(T_0;T_1,T_2)$ represents the coupon rate over the period $T_1,T_2]$, but where the size of the coupon rate itself is known at time T_0.

Once the principle of the determination of (the linear rate) $R(T_0;T_1,T_2)$ is known, it can be specified how the price of this cash flow can be determined.

Assume that $R(T_0;T_1,T_2)$ is defined as the yield at time T_0, of a zero-coupon bond expiring at time τ, then this means that $R(T_0;T_1,T_2)$, according to formula 47, can be written as follows (here shown in the one-dimensional case):

$$R(T_0;T_1,T_2) = R^F(T_0,T_0,\tau) = R^F(t,T_0,\tau) + \int_t^{T_0} \frac{\sigma_P(s,\tau) - \sigma_P(s,T_0)}{\tau - T_0} d\tilde{W}^{Q_{T_0}} + \frac{1}{2}\int_t^{T_0} \frac{(\sigma_P(s,\tau) - \sigma_P(s,T_0))^2}{\tau - T_0} ds \qquad \textbf{(100)}$$

Where a T_0 -period zero-coupon bond has been used as numerator.

Under this assumption regarding the determination of the coupon rate $R(T_0;T_1,T_2)$, this means that $P_V(t,T_2)$ can be formulated as follows:

$$\begin{aligned} P_V(t,T_2) &= P(t,T_2)[H + H(T_2 - T_1)E^{Q_{T_2}}[R(T_0;T_1,T_2)]] \\ &= P(t,T_2)\left[H + H(T_2 - T_1)\left[R^F(t,T_0,\tau) - \int_t^{T_0} \frac{[\sigma_P(s,\tau) - \sigma_P(s,T_0)]\sigma_P(s,T_2)}{\tau - T_0} ds\right]\right] \end{aligned} \qquad \textbf{(101)}$$

If we now make a generalization and consider a floating-rate coupon bond with a sequence of payments T_j, for j = {1,2,.....,n}, where the coupon rate that determines the interest payments over the individual periods j, is defined/known at time T_j - d, and where the yield that determines the coupon is the zero-coupon rate of the zero-coupon bonds expiring at time τ_j, for $(T_j - d) < \tau_j$, then this means that the price can be written as:

$$\begin{aligned} P_V^k(t) &= \sum_{j=1}^{n} P(t,T_j)[afd_j + (rg_j + afd_j)(T_j - T_{j-1})E^{Q_{T_j}}[R(T_{j-d};T_{j-1},T_j)]] \\ &= \sum_{j=1}^{n} P(t,T_j)\left[afd_j + (rg_j + afd_j)(T_j - T_{j-1})\left[R^F(t,T_{j-d},\tau_j) - \int_t^{T_{j-d}} \frac{[\sigma_P(s,\tau_j) - \sigma_P(s,T_{j-d})]\sigma_P(s,T_j)}{\tau_j - T_{j-d}} ds\right]\right] \end{aligned} \qquad \textbf{(102)}$$

Where afd_j and rg_j are the repayment and the balance, respectively, at time j.

In addition, this expression can be easily expanded so that the coupon rate that is defined/known at time T_{j-d}, **is not only** defined by the yield at time T_{j-d} of a zero-coupon bond expiring at time τ_j, but on the contrary as an arithmetical average of expected zero-coupon yields observed in a pre-defined period before time T_{j-d} on a zero-coupon bond with a fixed term to maturity[18]. However, rewriting formula 102 will not enhance the information value, and consequently it will not be made here since the notation ends up being rather complicated.

[18] Thus, it can be seen that this expression can be used to price the floating-rate government bonds listed on the Copenhagen Stock Exchange, and can therefore be considered an extension of the deterministic model from Madsen (1991).

6.8 Options on coupon bonds

In connection with the derivation of the semi-analytical price expression of options written on coupon bonds in section 6.2, it was stressed that the functional form given in formulas 60 and 61 would be analyzed in more detail, which is precisely what this section will do.

However, let me first recapitulate this price expression, thus:

$$
\begin{aligned}
C(t,T^F) &= \sum_{j=1}^{n} F_j \Phi P(t,T_j) Q_{T^F}(\Phi E_j) - \Phi X P(t,T^F) Q_{T^F}(\Phi E) \\
E_j &= \sum_{k=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - M_k + Cov(M_j,M_k)\right] \geq X \\
E &= \sum_{k=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - M_k\right] \geq X \\
& \qquad \textit{for} \\
M_k &= \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)] d\tilde{W}^{Q_{T^F}}(s) \\
V_k^2 &= \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)]^2 ds \\
Cov(M_j,M_k) &= \int_t^{T^F} [\sigma_P(s,T_j) - \sigma_P(s,T^F)][\sigma_P(s,T_k) - \sigma_P(s,T^F)] ds
\end{aligned}
\tag{103}
$$

As appears from this formula, we have to be able to calculate the the probabilities $Q_{T^F}(E_j)$, which means that it is necessary to calculate the distribution of a mix of log-normally-distributed variables, which, as mentioned previously, is no trivial problem.

However, there is one case where this calculation is made much easier, namely when the stochastic variables are proportional.

Proposition no. 6

If the volatility structure $\sigma^F(t,T)$ is only a function of t and T, then the spot rate r(t) follows a Markov process under the probability measure Q. This is because the Markovian structure implies that the stochastic variables are proportional.

In this connection we have that the following two functional forms of the forward volatility structure are proportional, namely the absolute model and the exponential

model[19], thus:

$$\sigma^F(t,T) = \begin{cases} \sigma \\ \sigma e^{-\kappa(T-t)} \end{cases} \quad \textit{for } t < T \tag{104}$$

Proof:

From formula 15 and 16 it is known that the spot rate and forward rate process under the probability measure Q is defined as follows:

$$\begin{aligned} r^F(t,T) &= r^F(0,T) + \int_0^t \sigma^F(s,T)\sigma_P(s,T)ds + \int_0^t \sigma^F(s,T)d\tilde{W}(s) \\ r(t) &= r(0) + \int_0^t \sigma^F(s,t)\sigma_P(s,t)ds + \int_0^t \sigma^F(s,t)d\tilde{W}(s) \end{aligned} \tag{105}$$

According to proposition no. 4 it is known that:

$$\begin{aligned} r^F(t,T) &= E^{Q_T}[r(T)] \\ &= E^Q[r(T)] - \int_t^T \sigma^F(s,T)\sigma_P(s,T)ds \end{aligned} \tag{106}$$

So that $E^Q[r(T)] = b(t,T)r(t) + a(t,T)$*, for b(t,T) and a(t,T) being deterministic functions that are only time-dependent, where this is precisely the Markov property if the process is Gaussian.*

Thus, if the following relation applies, it can be concluded that the spot rate is a Markov process under the probability measure Q:

$$E^Q[r(T)] = b(t,T)r(t) + a(t,T) \tag{107}$$

By comparing the relations for the spot rate and the forward rate, respectively, in formula 105 it can be deduced that this condition is equivalent to:

$$\int_0^t \sigma^F(s,T)d\tilde{W} = b(t,T)\int_0^t \sigma^F(s,t)d\tilde{W} \tag{108}$$

---

[19] I wish to point out that the absolute model is the continuous time version of the Ho and Lee (1986) model and the exponential model is identical to the volatility structure that is implicitly contained in the Vasicek (1977) model. Where it is moreover a natural requirement that $\kappa > 0$, (which is also assumed in the Vasicek model) as, otherwise, the volatility structure would not be bounded.

For $b(t,T) = \dfrac{\sigma^F(t,T)}{\sigma^F(t,t)}$*, where this condition is met for the volatility structure defined as*[20]

$\sigma^F(t,T) = \begin{cases} \sigma \\ \sigma e^{-\kappa(T-t)} \end{cases}.$

Now I intend to show that the proportionality condition is met for these two functional forms of the volatility structure.

Please bear in mind that under the probability measure Q_{TF}*,* M_k *can be written in the following form:*

$$M_k = \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)] d\tilde{W}^{Q_{T^F}}(s) \tag{109}$$

Assuming that the volatility structure for the forward rates can be written as
$\sigma^F(\tau) = \sigma e^{-\kappa\tau} \Rightarrow \sigma_P(\tau) = \dfrac{\sigma}{\kappa}[1 - e^{-\kappa\tau}]$*, this means that formula 109 will look as follows:*

$$\begin{aligned} M_k &= \frac{\sigma}{\kappa}\int_t^{T^F} [e^{-\kappa(T^F - s)} - e^{-\kappa(T_k - s)}] d\tilde{W}^{Q_{T^F}}(s) \\ &= \frac{\sigma}{\kappa}\int_t^{T^F} e^{\kappa s}[e^{-\kappa T^F} - e^{-\kappa T_k}] d\tilde{W}^{Q_{T^F}}(s) \\ &= \frac{\sigma}{\kappa}[e^{-\kappa T^F} - e^{-\kappa T_k}]\int_t^{T^F} e^{\kappa s} d\tilde{W}^{Q_{T^F}}(s) \end{aligned} \tag{110}$$

Which completes the argument, as the proportionality property appears from it[21]*.*

Q.E.D.

Knowing that these two functional forms of the volatility structure fulfil the proportionality

[20] It is with respect to this easy to see that the Markov property also is met by the following general formulations of the mean-reversion parameter κ, namely $\kappa_t = \int_t^T \kappa_s ds$.

[21] If the volatility structure is of the form $\sigma^F(\tau) = \sigma \rightarrow \sigma_P(\tau) = \sigma\tau$, it can be shown that M_k can be written in the following form: $M_k = \sigma[T_k - T^F]\int_t^{T^F} dW^{Q_{T^F}}(s)$, which proves the proportionality property.

condition, and since we know that it is possible to calculate the probability for a mix of log-normal distributions when the stochastic variables are proportional, I intend to construct a more appropriate expression of the price of an option written on a coupon bond.

Proposition no. 7

Given that the volatility structure is defined as in formula 104, the price of an option written on a coupon bond can be written as[22]:

$$
\begin{gathered}
C(t,T^F) = \sum_{j=1}^{n} \Phi F_j P(t,T_j) N(\Phi d_j) - \Phi X P(t,T^F) N(\Phi d) \\
\textit{for} \\
d_j = d + V_j \\
X = \sum_{j=1}^{n} F_j \frac{P(t,T_j)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_j^2 - V_j d\right] \\
V_j^2 = \int_t^{T^F} [\sigma_P(s,T_j) - \sigma_P(s,T^F)]^2 ds
\end{gathered} \tag{111}
$$

The option price can alternatively be written as the sum of a number of options on zero-coupon bonds with exercise price given by:

$$X_j = \frac{P(t,T_j)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_j^2 - V_j d\right] \tag{112}$$

Proof:

From formula 103 it is known that M_k is defined as:

$$M_k = \int_t^{T^F} [\sigma_P(s,T_k) - \sigma_P(s,T^F)] d\tilde{W}^{Q_{T^F}} \tag{113}$$

Due to the proportionality property, which is fulfilled for the absolute and the exponential models, it is known that this expression can be rewritten as:

$$M_k = V_k M_s \tag{114}$$

for V_K defined as in formula 103 and where M_s is a standardized normally distributed

---

[22] At this point I wish to stress that in their December (1993) paper Karoui, Myneni and Viswanathan derived a corresponding expression of the price of an option written on a coupon bond; however, there is a typo in their formula as they have defined d_j as $d + V_j^2$, which, as will appear from the proof, is incorrect.

variable, i.e. $M_s \sim N(0,1)$.

Rewriting the formula indicates that the event E can be expressed as:

$$\begin{aligned} E &= \sum_{j=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - M_k\right] \geq X \\ &= \sum_{j=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - V_k M_s\right] \geq X \end{aligned} \qquad \textbf{(115)}$$

As the left-hand side in formula 115 is a declining function of m_s *(for* m_s *being the mean value) this means that this expression can be written as:*

$$E = M_s \leq d \qquad \textbf{(116)}$$

Which means that there exists a critical interest rate d, for which it applies that for $d^* < d$ *then* $\sum_{j=1}^{F_k} \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - V_k d^*\right] > X$*, and conversely if* $d^* > d$.

This indicates that $Q_{T^F}(E) = N(d)$*, for d being the solution to the following equation:*

$$X = \sum_{j=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - V_k d\right] \qquad \textbf{(117)}$$

As concerns the event E_j*, it can be expressed as follows:*

$$\begin{aligned} E_j &= \sum_{j=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - M_k + Cov(M_j, M_K)\right] \geq X \\ &= \sum_{j=1}^{n} F_k \frac{P(t,T_k)}{P(t,T^F)} \exp\left[-\frac{1}{2}V_k^2 - V_k[M_s - V_j]\right] \geq X \\ & \text{as} \\ & Cov(M_j, M_k) = V_j V_k \end{aligned} \qquad \textbf{(118)}$$

Where this means that the event E_j *can be written as:*

$$E_j = M_s \leq d + V_j \qquad \textbf{(119)}$$

Where this completes the proof, as formulas 111 and 112 can be directly deduced.

Q.E.D.

Formula 112 can be seen to be identical to Jamshidian's (1989) bond option expression, if the forward volatility structure is defined as: $\sigma^F(t,T) = \sigma e^{-\kappa(T-t)}$.

Proposition no. 7 deserves an additional comment, as this is, together with Karoui, Myneni and Viswanathan, the first time in the literature that it has been possible, in a traditional term structure framework, to derive an analytical expression of the price of an option written on a coupon bond in a generalized Gaussian setting[23]. However, this is subject to the limitation that the volatility structure must fulfil the Markov condition, ie that it complies with the proportionality property and that the volatility structure is a function of time only.

However, it is possible to derive a more efficient analytical expression for the price of an option written on a coupon bond. The derivation is based on the relation between prices of options written on zero-coupon bonds and options written on the corresponding yields. Using this relation, the price of an option written on a coupon bond can be expressed as:

$$C(t,T^F) = \sum_{j=1}^{n} \Phi F_j P(t,T_j) N(\Phi d_1) - \Phi X P(t,T^F) N(\Phi(d_1 + V))$$

$$for$$

$$d_1 = -\frac{\ln\left(\dfrac{\sum_{j=1}^{n} F_j P(t,T_j)}{P(t,T^F)X}\right)}{V} - \frac{1}{2}V \qquad \textbf{(120)}$$

$$V = \sum_{j=1}^{n}\left(\frac{F_j P(t,T_j)\sqrt{\sum_{i=1}^{m}\int_t^{T^F}[\sigma_P(s,T_j;i) - \sigma_P(s,T^F;i)]^2 ds}}{\sum_{j=1}^{n} F_j P(t,T_j)}\right)$$

ere n=1 means that this expression will degenerate into the expression from formula 77 for a = 1 and $P(t,T) = aP(t,T^F)E^{Q_{T^F}}[R(T^F,T)]$.

This result is a theoretical extension of Jamshidian's (1989) one-factor framework for the pricing of options on coupon bonds and of the Karoui, Myneni and Viswanathan (1993) option expression from formula 111. This formula, which is shown in formula 120, is in fact rather general, whereas the two most significant propositions made at this point are that the interest process is normally distributed and that the volatility structures are only time-dependent and can thus be considered as being a fairly general framework for pricing options on coupon bonds.

[23] At any rate as far as is known.

6.9 Interest Rate Swaps[24] [25]

A plain vanilla fixed-for floating swap is an agreement in which one side agrees to pay a fixed rate of interest in exchange for receiving a floating rate of interest during the tenor (maturity) of the swap. The other counterparty to the swap agrees to pay floating and receive fixed. The two interest rates are applied to the swap's notional principal amount.

The fixed rate - called the swap rate - is set against a floating reference rate (for example London Interbank Offered Rate). The floating rate is reset several times over the life of the swap - usually every 3-6 month.

In the terminology of the swap market, the fixed rate payer is known as the buyer of the swap, and the floating rate payer is known as the seller. Furthermore the fixed rate is known as the price of the swap. Since the interest payments are computed on the same notional principal for both the counterparties, there is no exchange of principal between them at maturity. As is natural the fixed rate in the swap is fixed at initiation for the entire life of the swap.

The most common version of the fixed-for-floating plain vanilla swap is one where payments are made semi-annually and the floating rate is the 6-month US$ LIBOR. At the initiation date, the market value of the swap is set to zero.

If we disregard credit-risk, then a long position in a swap contract can be thought of as a long position in a floating rate note and a short position in a fixed rate. Since the floater is set equal to the market rate for the entire term of the floater, it is assumed that the hypothetical floater would trade at par on a reset date. Hence, the fixed rate on the swap must be specified so that the hypothetical fixed rate note would also trade at par - which then makes the net present value of the swap equal to zero.

This leads us to the following conclusion:

In the absence of credit-risk, the arbitrage-free fixed swap rate at initiation should equal the yield on a par coupon bond that makes fixed payments on the same dates as the floating leg of the swap.

Let us here consider a forward start swap settled in arrears with a notional amount equal to H. The payments are made at times T_j, for j = {1,2,....,J} and $t < T_1 < T_2 < ... < T_J$. We asume equal distance between payment dates, ie $\delta = T_j - T_{j-1}$, for all j ={1,2,...J}. The floating rate $L(T_{j-1},T_j)$ - which is set at time T_{j-1} and received at time T_j - can be derived from the price of a zero-coupon bond over that period, that is:

[24] The basic description of an interest swap is from Svensson (1998).

[25] Here I will disregard swaps containing some sort of option element, and cases where the floating rate is found as some sort of average over some predefined period. Thus, I will among others disregard "extendable"-and "puttable"-swaps. More precisely I will only be considering the plain vanilla swap.

$$L(T_{j-1},T_j) = \frac{1}{\delta}\left(\frac{1}{P(T_{j-1},T_j)} - 1\right) \quad \textbf{(121)}$$

From this we can deduce that the cash-flow at any date T_j, for j = [1,2,..,J], of a payer swap is $L(T_{j-1},T_j)\delta H$ and $-s\delta H$, where s is the fixed rate. The cash-flow of a receiver swap has the same size, but is of opposite signs.

Considering a forward swap we will denote the dates (T_0, T_1,....,T_{J-1}) - reset dates and the dates
(T_1, T_2,....,T_J) - settlement dates, where $0 \leq t \leq T_0$. We will refer to the start date of the swap as T_0.

We are now ready to specify how the value of a forward start swap (at time t) can be expressed, ie:

$$F^{Swap}(t) = \sum_{j=1}^{J} E^{Q}\left[\frac{P(0,T_j)}{P(0,t)}\eta[L(T_{j-1},T_j) - s]H\delta\right] = \sum_{j=1}^{J} E^{Q}\left[P(t,T_j)\eta[\frac{H}{P(T_{j-1},T_j)} - \tilde{s}]\right] \quad \textbf{(122)}$$

Where $\tilde{s} = H + s\delta H$.

Doing a bit of algebra we can rewrite formula 122 as (for H = 1):

$$F^{Swap}(t) = \eta\left[P(t,T_0) - \sum_{j=1}^{J} c_j P(t,T_j)\right] \quad \textbf{(123)}$$

for every t = [0,T], where we also have that $c_j = sH\delta$, for j = [1,2,...,J-1] and where $c_J = H(1 + s\delta)$. We have that η = [1,-1], if $\eta = 1$ then it is a payer swap and if $\eta = -1$ then it is a receiver swap.

That is a forward payer swap settled in arrears can be viewed as a contract to deliver a specific coupon bond and to receive at the same time a zero-coupon bond. From this we can deduce that a forward receiver swap settled in arrears can be viewed as a contract to receive a specific coupon bond and to deliver at the same time a zero-coupon bond

In many cases the swap is settled not in arrears but in advance - that is the reset date is equal to the settlement date. However, in this case the discounting method differs from country to country - in general for some minor changes formula 123 is still valid.

We have mentioned a couple of times that a swap at initiation has a value equal to zero. This important feature leads to the following definition:

Definition no. 1

The forward swap rate at time t for time T_0 is given by:

$$s(t,T_0) = \frac{P(t,T_0) - P(t,T_J)}{\sum_{j=1}^{J} P(t,T_j)\delta} \qquad \textbf{(124)}$$

which means that the forward swap rate is defined as the fixed rate that ensures that the value of the forward start swap is 0 (zero). From formula 124 it can be seen that the swap rate at time t for time T_0 is the rate which guarantees that the discounted value of the fixed leg is identical to the difference between the price of the zero coupon bond with maturity at date T_0 and the price of the zero coupon bond that matures at time T_J (the expiry date of the swap).

It can be deduced from formula 124, that: $s(T_0,T_0) = \dfrac{1 - P(T_0,T_J)}{\sum_{j=1}^{J} P(T_0,T_j)\delta}$**.**

If we now assume that the swap only has one leg, that is J = 1, formula 124 simplifies into:

$$s(t,T_0) = \frac{P(t,T_0) - P(t,T)}{P(t,T)(T - T_0)} \qquad \textbf{(125)}$$

From this it follows that for t=0, we have:

$$s(0,T_0) \approx \frac{TR(0,T) - T_0R(0,T_0)}{T - T_0} = f(0,T_0,T) \qquad \textbf{(126)}$$

where $f(0,T_0,T)$ is the forward interest rate determined by a forward rate agreement. From this it follows that the swap rate does not coincide with the forward rate determined from a forward rate agreement.

6.10 Caps and Floors

Assume that R^{Cap} is the Cap rate, H is the principal (the nominal amount), and interest payments are made at times T_j, for j = {1,2,....,J} and $t < T_1 < T_2 < ... < T_J$, then the price of the Cap settled in arrears at time t can be written in the following form:

$$P^{Caps}(t) = \sum_{j=0}^{J-1} E^{Q}\left[\left(\exp - \int_t^{T_j} r^F(t,s)ds\right) \frac{H(T_{j+1} - T_j)}{1 + (T_{j+1} - T_j)L(T_j,T_{j+1})} \max[L(T_j,T_{j+1}) - R^{Caps};0]\right] \qquad \textbf{(127)}$$

This expression is to be understood in the sense that at time T_j the issuer of the Cap must pay the difference between the floating interest rate $L(T_j,T_{j+1})$, which covers the period $[T_j,T_{j+1}]$, and the capped interest rate R^{Caps}, times the principal H.

As opposed to a Cap, a Floor is not a maximum rate that the Cap investor may have to pay on a floating-rate loan at time T_{j+1}, but on the other hand a minimum rate that can be demanded at time T_{j+1}, i.e. in formula 127 it will not be $\max[L(T_j,T_{j+1}) - R^{Caps};0]$ but instead $\max[R^{Caps} - L(T_j,T_{j+1});0]$.

As $L(T_j,T_{j+1}) = \frac{1}{T_{j+1} - T_j}\left(\frac{1}{P(T_j,T_{j+1})} - 1\right)$, this means that the price of the Cap defined by formula 127 can be rewritten as:

$$\begin{gathered} P^{Caps}(t) = \sum_{j=0}^{J-1}\left(\exp - \int_t^{T_{j+1}} r^F(t,s)ds\right) E^{Q_{T_j}}[\max[H - \tilde{R}^{Caps};0]] \\ \textit{for} \\ \tilde{R}^{Caps} = H + H(T_{j+1} - T_j)R^{Caps} \end{gathered} \qquad \textbf{(128)}$$

Thus, this means that the price of a Cap can be determined as the sum of the prices of a number of put options on zero-coupon bonds, where each of these options is popularly known as Caplets. In addition, it applies that the exercise price of the put option is H and that the individual Caplets have a face value of $\tilde{R}^{Caps}$.

Now that we know the previously derived price of a call option on a zero-coupon bond from formula 72, the price of a Cap can be written as:

$$\begin{gathered} P^{Caps}(t) = \sum_{j=0}^{J-1}\left[P(t,T_j)\Phi HN((-d_1 + V_j)\Phi) - P(t,T_{j+1})\Phi\tilde{R}_j^{Caps}N(-d_1\Phi)\right] \\ \textit{for} \\ d_1 = \frac{\ln\left(\frac{P(t,T_{j+1})\tilde{R}^{Caps}}{P(t,T_j)H}\right)}{V_j} + \frac{1}{2}V_j \\ V_j = \sqrt{\int_t^{T_j}[\sigma_P(s,T_{j+1}) - \sigma_P(s,T_j)]^2 ds} \\ \tilde{R}_j^{Caps} = H + H(T_{j+1} - T_j)R^{Caps} \end{gathered} \qquad \textbf{(129)}$$

The price of a Floor can then be found by fixing Φ= -1.

From equation 122 and 128 we can derive the put-call parity relationship between the prices of caps and floors, more precisely we have:

$$P^{Caps}(t) = P^{Floor}(t) + P^{Swap}(t) \tag{130}$$

if we assume that the cap and floor have the same strike-price R^{cap}, that the swap is an agreement to receive floating and pay the fixed rate (a payer swap) R^{cap} with no exchange of payments on the reset dates, and lastly, that all three instruments have the same life and the same frequency of payments.

6.11 Swaptions[26]

Swaptions are options on a swap. More precisely, a swaption can be regarded as an option on the fixed leg with a strike price equal to par. The reason for that is that the floating leg at the time of contracting in a (traditional) swap is always equal to par. As the fixed leg can be understood as being constructed by a portfolio of zero coupon bonds, this means, which will become apparent from the following, that pricing a swaption is identical to pricing an option on a coupon bond.

Form this it can be concluded that if a swaption gives the investor the right to pay fixed and receive variable, it is a put-option on the fixed leg with a strike price equal to the nominal amount. Alternatively, if a swaption gives the investor the right to pay variable and receive fixed, it is a call-option on the fixed leg with a strike price equal to the nominal amount.

It can also be shown that a forward start swap is identical to the difference between a call-swaption (payer swaption) and a put-swaption (a receiver swaption).

It is assumed that there exists an option (payer swaption) with an exercise date at time T^F, written on the swap defined by formula 123, for $t < T^F = T_0 < T_1$.

The price of this payer swaption can be written in the following form:

$$C(t,T^F) = E^Q\left[\exp\left(-\int_t^{T^F} r^F(t,s)ds\right)\max\left[\sum_{j=1}^{n} P(T^F,T_j)[S_r(T^F,T^F) - X](T_j - T_{j-1});0\right]\right] \tag{131}$$

It appears from this formula that this expression is identical to formula 19, apart from the fact that $P^k(T^F)$ has been replaced by $S_r(T^F,T^F)$, where $S_r(T^F,T^F)$ is the forward swap rate at time T^F.

The forward swap rate at time t for time T^F is given by (see Definition no. 1):

[26] When using the word swap we are implicitly assuming that it is swap settled in arrears.

$$S_r(t,T^F) = \frac{P(t,T^F) - P(t,T_J)}{\sum_{j=1}^{J} P(t,T_j)(T_j - T_{j-1})} \tag{132}$$

which means that the forward swap rate is defined as the fixed rate that ensures that the value of the forward start swap is 0 (zero). It can be deduced from formula 132 that

$S_r(T^F,T^F) = \dfrac{1 - P(T^F,T_J)}{\sum_{j=1}^{J} P(T^F,T_j)(T_j - T_{j-1})}$ which means that formula 131 can rewritten as:

$$\begin{gathered} C(t,T^F) = \exp\left(-\int_t^{T^F} r^F(t,s)ds\right) E^{Q_{T^F}}\left[\max\left[1 - \sum_{j=1}^{J} X(j)P(T^F,T_j);0\right]\right] \\ for \\ X(j) = X(T_j - T_{j-1})\ for\ j < J\ og\ X(j) = 1 + X(T_j - T_{j-1})\ for\ j = J \end{gathered} \tag{133}$$

Where it follows that the price is given by a put-option on the fixed leg with a strike price equal to the nominal amount.

Thus, the definition of the option price expression follows the principle used when the semi-analytical price expression of options on coupon bonds was derived, and in this connection yields the following result:

$$\begin{gathered} C(t,T^F) = \sum_{j=1}^{J} \Phi P(t,T_j)X(j)Q_{T^F}(\Phi E_j) - \Phi P(t,T^F)Q_{T^F}(\Phi E) \\ for \\ E_j = \sum_{k=1}^{J} X(k)\frac{P(t,T_k)}{P(t,T^F)}\exp\left[-\frac{1}{2}V_k^2 - M_k + Cov(M_j,M_k)\right] \geq 1 \\ E = \sum_{k=1}^{J} X(k)\frac{P(t,T_k)}{P(t,T^F)}\exp\left[-\frac{1}{2}V_k^2 - M_k\right] \geq 1 \\ for \\ M_k = \int_t^{T^F}[\sigma_P(s,T_k) - \sigma_P(s,T^F)]d\tilde{W}^{Q_{T^F}}(s) \\ V_k^2 = \int_t^{T^F}[\sigma_P(s,T_k) - \sigma_P(s,T^F)]^2 ds \\ Cov(M_j,M_k) = \int_t^{T^F}[\sigma_P(s,T_j) - \sigma_P(s,T^F)][\sigma_P(s,T_k) - \sigma_P(s,T^F)]ds \end{gathered} \tag{134}$$

The solution to formula 133 follows by defining Φ= -1.

If we now consider a swaption on a one-period swap, i.e. J = 1, this means that formula 134, by using the principles applied when the price expression for options on zero-coupon bonds was derived, can be reduced to:

$$C(t,T^F) = \Phi P(t,T_j)X(j)N(\Phi d_1) - \Phi P(t,T^F)N(\Phi(d_1 - V))$$
$$for$$
$$d_1 = \frac{\ln\left(\dfrac{P(t,T_j)X(j)}{P(t,T^F)}\right)}{V} + \frac{1}{2}V \qquad \textbf{(135)}$$

where this expression gives the price of a payer swaption on a one-period swap. The value of a receiver swaption is easy derived, namely by defining Φ= -1.

It now follows that when we consider a payer swaption on a one-period swap, the expression is reduced to the price expression from formula 129 - for H = 1, $\tilde{R}_j^{Caps} = X(j)$ and J = 1.

This implies that a caplet is identical to a one-period payer swaption.

7. The Pricing of options on Coupon Bonds - an example

The pricing equation for the value of an option on a coupon bond represented by formula 120 has been derived by using the relationship between options prices on yield and bonds. For that reason the formula should also lend itself usuable for the pricing of options on coupon bonds in a multi-factor Markovian model - as we with respect to the valuation of options on yields managed to get a closed form solution in the multi-factor setting, see formula 80.

I do not claim to have shown (actually I have not been able to justify it mathematically) that the option pricing formula in equation 120 is a “true” closed form solution for the pricing of options on coupon bonds in a general multi-factor Markovian term structure setting - but the results presented in the following 3 tables does indicate that the formula might have some usefullness in this respect.

Another methodology with recently has been applied to the pricing of options on coupon bonds is the stochastic duration approach, see Wei (1997) and Munk (1998)[27]. In order to test the specification from formula 120 I have compared it to results obtained by Munk using the

[27] The general idea behind the stochastic duration approach is that the price of a European option on a coupon bond can be approximated by a multiple of the price of a European option on a zero-coupon bond with a time to maturity equal to the stochastic duration of the coupon bond.

stochastic duration approach on a 2-factor HJM model, more precisely he considers the following 2-factor HJM model:

$$\sigma_P(t,T;1) = \sigma_1(T - t)$$
$$\sigma_P(t,T;2) = \frac{\sigma_2}{\kappa}[1 - e^{-\kappa(T - t)}] \tag{136}$$

In line with Wei and Munk we assume that the initial yield-curve is generated by the CIR model, with the following parameters; mean-reversion = 0.25, unconditional mean = 0.085, volatility 0.05 and spot-rate 8%. Furthermore we specify the parameters for the volatility process in formula 136 as: $\sigma_1 = \sigma_2 = 0.02$ and $\kappa = 0.5$.

In table 1 we have shown the price of both a 4-Month call-option and a put-option on a 5-Year bullit bond with an annual coupon of 8%. In this connection we have that the spot-price is 97.9788 and the forward-price is 100.6334:

Table 1: Prices for options on a 5-Year Coupon Bond

Exercise-Price	**Call-Option price Munks[28] from his table 3, page 21**	**Call-Option price - formula 120**	**Put-Option price - formula 120**
95	5.76101	5.76232	0.27750
96	4.93245	4.93411	0.42292
97	4.15706	4.15905	0.62148
98	3.44428	3.44657	0.88262
99	2.80201	2.80454	1.21420
100	2.23573	2.23839	1.62168
101	1.74791	1.75060	2.10751
102	1.33784	1.34045	2.67098
103	1.00178	1.00421	3.30836
104	0.73344	0.73563	4.01340
105	0.52481	0.52670	4.77809
106	0.36689	0.36847	5.59349

As is obvious from table 1 the result I have obtained for a call-option using formula 120 is

[28] Munk (1998) only calculate call-option prices.

close to the results reported by Munk. Munk test his result using Monte Carlo simulation, I will do the same - though I will simulate under the T-adjusted probability measure with is more efficient.

More precisely the simulation will be performed under the T^F-adjusted probability measure - where T^F symbolises the expiry-date of the option (that is T^F = 4-Month), ie:

$$P(T^F,T) = \frac{P(0,T)}{P(0,T^F)}\exp\left[-\frac{1}{2}\sum_{i=1}^{2}\int_0^{T^F}[\sigma_P(s,T;i) - \sigma_P(s,T^F;i)]^2 ds - \sum_{i=1}^{2}\int_0^{T^F}[\sigma_P(s,T;i) - \sigma_P(s,T^F;i)]dW^{Q_{T^F}i}(s)\right] \quad \textbf{(137)}$$

Where $\sigma_p(t,T;i)$ - the bond-price volatility - for each i = [1,2] is defined in equation 136.

We have employed the following Monte Carlo simulation methods: Crude Monte Carlo, Antithetic Monte Carlo, Stratified Sampling and Empirical Martingale Simulation (EMS)[29].

We have performed B batches of N simulations, for B = 100 and N = 10000. Provided that B is sufficiently large, the Central Limit Theorem implies that the batch error is approximately Gaussian. The sampled variance can under this assumption be calculated as:

$$Var_B = \frac{1}{B(B-1)}\left(B\sum_{b=1}^{B}\hat{C}_b^2 - \left(\sum_{b=1}^{B}\hat{C}_b\right)^2\right) \quad \textbf{(138)}$$

Where $\hat{C}_b$ is the estimated/simulated option-price for the b'th batch-run.

Table 2: Simulated Call option prices in the 2-factor HJM model

Excercise Price	**Crude Monte Carlo**	**Std. Error[30]**	**Antithetic Monte Carlo**	**Std. Error**	**Stratified Samp-ling**	**Std. Error**	**Empirical Monte Carlo Simulation**	**Std. Error**
95	*5.75390*	*0.04356*	*5.75982*	*0.00770*	*5.75990*	*0.00014*	*5.75929*	*0.00863*
96	*4.93201*	*0.04067*	*4.93215*	*0.00941*	*4.93157*	*0.00012*	*4.93093*	*0.01027*
97	*4.15191*	*0.03972*	*4.15501*	*0.01147*	*4.15664*	*0.00013*	*4.15933*	*0.01146*
98	*3.45528*	*0.03874*	*3.44489*	*0.01381*	*3.44446*	*0.00013*	*3.44394*	*0.01208*

[29] See Appendix G in Madsen (1998) where these methods are briefly explained.

[30] Std. Error is calculated as the square root of the sampled variance specified in formula 138.

99	*2.80323*	*0.03231*	*2.80317*	*0.01342*	*2.80295*	*0.00015*	*2.80247*	*0.01593*
100	*2.23787*	*0.03070*	*2.23597*	*0.01523*	*2.23749*	*0.00014*	*2.23766*	*0.01411*
101	*1.74606*	*0.02669*	*1.74913*	*0.01652*	*1.75059*	*0.00015*	*1.75061*	*0.01395*
102	*1.33888*	*0.02559*	*1.34239*	*0.01386*	*1.34142*	*0.00014*	*1.34144*	*0.01539*
103	*1.00459*	*0.02013*	*1.00476*	*0.01191*	*1.00604*	*0.00013*	*1.00537*	*0.01342*
104	*0.73635*	*0.01746*	*0.73666*	*0.01192*	*0.73809*	*0.00013*	*0.73885*	*0.01310*
105	*0.52776*	*0.01342*	*0.52930*	*0.00864*	*0.52952*	*0.00013*	*0.52781*	*0.01062*
106	*0.36912*	*0.01418*	*0.37156*	*0.00877*	*0.37135*	*0.00013*	*0.37036*	*0.00937*

Table 3: Simulated Put option prices in the 2-factor HJM model

Excercise Price	**Crude Monte Carlo**	**Std. Error**[31]	**Antithetic Monte Carlo**	**Std. Error**	**Stratified Samp-ling**	**Std. Error**	**Empirical Monte Carlo Simulation**	**Std. Error**
95	*0.27487*	*0.00847*	*0.27393*	*0.00748*	*0.27505*	*0.00008*	*0.27361*	*0.00914*
96	*0.42128*	*0.01238*	*0.42081*	*0.00792*	*0.42038*	*0.00009*	*0.42018*	*0.00992*
97	*0.61874*	*0.01476*	*0.61796*	*0.01000*	*0.61904*	*0.00008*	*0.61997*	*0.01199*
98	*0.87845*	*0.01970*	*0.87903*	*0.01143*	*0.88053*	*0.00011*	*0.87806*	*0.01453*
99	*1.21462*	*0.02189*	*1.21184*	*0.01360*	*1.21259*	*0.00008*	*1.21390*	*0.01326*
100	*1.62122*	*0.02615*	*1.62109*	*0.01163*	*1.62074*	*0.00009*	*1.62388*	*0.01364*
101	*2.10785*	*0.02825*	*2.10804*	*0.01249*	*2.10748*	*0.00009*	*2.10657*	*0.01393*
102	*2.66944*	*0.02909*	*2.67332*	*0.01299*	*2.67194*	*0.00009*	*2.67276*	*0.01437*
103	*3.30842*	*0.03238*	*3.30789*	*0.01162*	*3.31020*	*0.00009*	*3.30911*	*0.01398*
104	*4.01653*	*0.03478*	*4.01632*	*0.01013*	*4.01586*	*0.00010*	*4.01635*	*0.01036*
105	*4.77676*	*0.03653*	*4.78026*	*0.00886*	*4.78090*	*0.00010*	*4.78107*	*0.01136*
106	*5.59129*	*0.04049*	*5.59527*	*0.00732*	*5.59636*	*0.00010*	*5.59768*	*0.00949*

[31] Std. Error is calculated as the square root of the sampled variance specified in formula 138.

From these tables we conclude that there is evidence that the general formula for the pricing of options on coupon bonds from formula 120 even can be used in a multi-factor Markovian yield-curve model. Ofcourse this is not a formal test (proof) of the formula - but the results are promising.

8. Conclusion

The most important result in this working paper is the construction of a multi-dimensional Gaussian interest-rate term structure model, where, based on a construction of equivalent martingale measures and a suitable selection of numerators, it was shown that it was possible to derive analytical expressions for a wide range of derived instruments.

The instruments for which analytical expressions were derived in this connection were forward contracts, futures contracts, options on zero-coupon bonds, options on interest rates (including options on the slope and curvature of the term structure of interest rates), caps and floors, options on both forward contracts and futures contracts, options on CIBOR futures, including options on FRAs, the pricing of floating-rate bonds, swaps, swaptions and, finally, options on coupon bonds.

As regards the price expression for options on coupon bonds, a generalization was made of the Karoui, Myneni and Viswanathan (1993) model. The analytical expression derived here is namely a "real" analytical expression as opposed to Karoui, Myneni and Viswanathan which must be considered to be a semi-analytical expression.

Using numerical tests we even managed to show that the new “true” closed form formula for the pricing of options on coupon bonds even seems to work in a general multi-factor Markovian HJM framework.

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Appendix A

In line with Karoui, Myneni and Viswanathan (1993) and Geman, Karoui and Rochet (1993) I will use the change of numeraire as the fundamental tool when analysing the properties of the processes for interest rates and bond prices.

Definition no. A.1

A numerator is a price-process N_t which is positive for all t ε [0,τ].

Because of "the numeraire invariance theorem" there will be no arbitrage opportunities for the price process P(t,T), if and only if there are no arbitrage opportunities for the normalized price process $\frac{P(t,T)}{N_t}$. If the normalized price process is a martingale then there are no arbitrage opportunities. Furthermore we have that the martingale property is valid if N_t is a regular numerator.

From this we can deduce that if the normalizsed price process allows for the existence of an equivalent martingale measure, this is both a sufficient and neccesary condition for no arbitrage opportunities.

To ensure the arbitrage-free properties of the bond market, we therefore need to examine the existence of a martingale measure for a suitable choice of numeraire. In our setup this means that we can either use the bond price P(t,T) or the money market account M(t).

In section 3 in the main text we are deriving the risk-neutral process for the forward-rates by postulating the process for bond prices under the Q-probability measure - that is we are deriving the forward-rate process the other way around. Let me here do it the traditional way when it is the Heath, Jarrow and Morton framework that is being considered - that is start by specifying the process for the forward-rates.

For every fixed T ≤ τ, the dynamics of the instantaneous forward-rates under the original probability measure P are given by[32]:

$$r^F(t,T) = r^F(0,T) + \int_0^t \mu^F(s,T)ds + \sum_{i=1}^{m} \int_0^t \sigma^F(s,T;i)dW_i(s) \qquad \textbf{(139)}$$

In principle it is no problem to assume that both the drift-and diffusion-coefficients are functions of the forward-rates - that is the forward-rates are lognormal. We will however not be considering this case, for the interested reader we refer to Miltersen (1994). One thing is interesting to mention however - namely that it can be shown that under the risk-neutral probability measure the SDE does not permit a global solution, that is, its local solution explodes in a finite time, see Morton (1989) and Miltersen (1994).

[32] We assume that the coefficients μ^F and σ^F are bounded function.

From formula 1 we can deduce that the spot-rate process under the P-probability measure can be written as:

$$r(t) = r^F(0,t) + \int_0^t \mu^F(s,t)ds + \sum_{i=1}^{m} \int_0^t \sigma^F(s,t;i)dW_i(s) \tag{140}$$

From formula 1 and the relationship between forward-rates and bond-prices from equation 1 in the main text, we can derive the process for the bond-price under the P-probability measure.

Proposition A.1

The dynamics of the bond-price P(t,T) are determined by the following SDE:

$$dP(t,T) = [r(t) + b(t,T)]P(t,T)dt - \sum_{i=1}^{m} \sigma_P(t,T;i)P(t,T)dW_i(t)$$
$$where$$
$$b(t,T) = -\int_t^T \mu^F(t,v)dv + \frac{1}{2}\sum_{i=1}^{m} \sigma_P^2(t,T;i) \tag{141}$$
$$and$$
$$\sigma_P(t,T;i) = \int_t^T \sigma^F(t,v;i)dv$$

Proof:

See lemma 13.1.1 in Musiela and Rutkowski (1997 page 306).

Qed.

Before continuing let me first introduce Girsanovs Theorem, as we will using this in the derivation.

Definition no. A.2 The m-dimensional Girsanovs Theorem

- **Let $\theta(t) = [\theta(1,t), \theta(2,t),\ldots\ldots, \theta(m,t)]$, for $0 \le t \le T \le \tau$, be an m-dimensional adapted process.**

For $0 \le t \le T \le \tau$, define:

$$\tilde{W}_i = W_i - \int_0^t \theta(i;s)ds \ ; for\ i\ \varepsilon\ [1,2\ldots,m]$$
$$\rho(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds \right) \tag{142}$$

Use ρ(T) as a Radon-Nikodym derivative to define a new probability measure, ie:

$$Q(A) = \int_A \rho(T)dP \qquad \text{for all } A \;\varepsilon\; F, \tag{143}$$

which means that $E^Q[V_t] = E[\rho(T)V_t]$**, for every random variable** V_t**. Then under Q,** $\tilde{W}(t) = [\tilde{W}_1(t), \tilde{W}_2(t),....., \tilde{W}_m(t)]$ **is an m-dimensional Wiener process.**

The two probability measures Q and P are now said to be equivalent if P(A) = 0 and Q(A) = 0 for every event A ε F.

We are looking for a condition ensuring the absence of arbitrage opportunities for all bonds of different maturities. Let us introduce the normalised price-process defined by:

$$F(t,T_1,T) = \frac{P(t,T)}{P(t,T_1)} \tag{144}$$

which can be recognized as the forward-price.

Given the dynamic for the bond-price in formula 3, we can deduce that the SDE for the forward-price can be written as:

$$\begin{aligned} dF(t,T_1,T) &= \left[\mu^F(t,T) - \mu^F(t,T_1) - \sum_{i=1}^{m} \tilde{\mu}(t,T;i)\right]F(t,T_1,T)dt \\ &\quad + \sum_{i=1}^{m} [\sigma_P(t,T;i) - \sigma_P(t,T_1;i)]dW_i(t) \\ &\qquad \textit{where} \\ \tilde{\mu}(t,T;i) &= \sigma_P(t,T_1;i)[\sigma_P(t,T;i) - \sigma_P(t,T_1;i)] \end{aligned} \tag{145}$$

If we instead consider the relative price process

Definition no. 3 The m-dimensional Martingale representation Theorem.

- **Let** $\theta(t) = [\theta(1,t), \theta(2,t),....., \theta(m,t)]$**, for 0 ≤ t ≤ T ≤τ, be a m-dimensional adapted process and define Q and** $\tilde{W}$ **as in the m-dimensional Girsanovs theorem from definition no. 2.**
- **Let B(t), for 0 ≤ t ≤ T ≤τ, be a P-martingale, starting at zero.**
- **Let** $\tilde{B}(t)$**, for 0 ≤ t ≤ T ≤τ, be a Q-martingale, starting at zero.**

Then B(t), 0 ≤ t ≤ T ≤τ, is a stochastic integral with respect to W and $\tilde{B}(t)$**, 0 ≤ t ≤ T ≤τ, is a stochastic integral with respect to** $\tilde{W}$**. In other words there exist a m-dimensional adapted process** $\varphi(t) = [\varphi(1,t), \varphi(2,t),....., \varphi(m,t)]$ **and** $\tilde{\varphi}(t) = [\tilde{\varphi}(1,t), \tilde{\varphi}(2,t),....., \tilde{\varphi}(m,t)]$**,**

such that: $B(t) = \int_0^t \sum_{i=1}^m \varphi_i(s) dW_i(s),\ and\ \tilde{B}(t) = \int_0^t \sum_{i=1}^m \tilde{\varphi}_i(s) d\tilde{W}_i(s)$**.**

Let now V_T be a contingent claim, ie an F_T-measurable random variable representing the payoff at time T. Suppose that there exist a hedge which results in $X(T) = V_T$ almost surely[33].

The martingale property of $\frac{X(t)}{N_t}$ gives the risk-neutral valuation formula (for $M(t) = N_t$):

$$\frac{X(t)}{N_t} = E^Q\left[\frac{X(T)}{N_T}|F_t\right] = E^Q\left[\frac{V_T}{N_T}|F_t\right] \tag{146}$$

In particulary, the value of the contingent claims at time t is:

$$X(t) = N_t E^Q\left[\frac{V_T}{N_T}|F_t\right] \tag{147}$$

Since $N_0 = 1$, the initial value of the contingent claim is:

$$X(0) = E^Q\left[\frac{V_T}{N_T}\right] \tag{148}$$

In order to prove the above relations we need to access that $\frac{X(t)}{N_t}$ is a martingale under the risk-neutral probability measure Q.

Proposition no. 1

$\frac{X(t)}{N_t}$ **is a martingale under the risk-neutral probability measure Q.**

Proof:

See Musiela and Rutkowski (1997 section 13.1.3).

Qed.

As we from proposition no. 1 know that $\frac{P(t,T)}{N_t}$ is a martingale given T, we can conclude that

[33] Which probability 1.

$\dfrac{\dfrac{P(t,T)}{N_t}}{\dfrac{P(0,T)}{N_0}}$ is a martingale, which "starts" in 1. As $N_0 = 1$, we get the following expression for all $0 \leq t \leq T < \tau$:

$$E\left[\frac{P(t,T)}{N_t P(0,T)}\right] = 1 \tag{149}$$

Which means that $\dfrac{P(t,T)}{N_t P(0,T)}$ specifies a density function.

Now define a new probability measure Q_T:

$$\frac{dQ_T}{dQ} = \frac{1}{N_T P(0,T)} \tag{150}$$

Proposition no. 2

Let now Q be the risk-neutral probability measure - that is the probability measure that is associated with using the money-market account as numerator. Then the probability measure Q_T - defined by the density function from formula 15 with respect to Q - is the probability measure that is associated with using the T-period zero-coupon bond as numerator.

Proof:

Assume that $\dfrac{V_t}{N_t}$ *is a Q-martingale for* V_t *being the value of any contingent claim at time t.*

For $t < T_1 < T$ *we have:*

$$E^{Q_T}\left[\frac{V_{T_1}}{P(T_1,T)}|F_t\right] = \frac{E^Q\left[\dfrac{V_{T_1}}{P(T_1,T)}\dfrac{1}{N_T P(0,T)}|F_t\right]}{E^Q\left[\dfrac{1}{N_T P(0,T)}|F_t\right]}$$

$$= \frac{E^Q\left[\dfrac{V_{T_1}}{P(T_1,T)}E^Q\left[\dfrac{P(T,T)}{N_T P(0,T)}|F_{T_1}\right]|F_t\right]}{E^Q\left[\dfrac{1}{N_T P(0,T)}|F_t\right]} = \frac{E^Q\left[\dfrac{V_{T_1}}{P(T_1,T)}\dfrac{P(T_1,T)}{N_{T_1} P(0,T)}|F_t\right]}{E^Q\left[\dfrac{P(T,T)}{N_T P(0,T)}|F_t\right]} = \frac{\dfrac{V_t}{N_t}\dfrac{1}{P(0,T)}}{\dfrac{P(t,T)}{N_t P(0,T)}} = \frac{V_t}{P(t,T)} \tag{151}$$

In the proof we have used Bayers rule in the first line and the martingale property for arbitrage-free financial markets, see formula 12.

Qed.

The results from proposition no. 2 do not rely on a specific assumption imposed on the dynamics of bond prices. However, in certain cases an explicit solution for the density in formula 15 is avaiable - which for example is the case for the bond price process we will consider in section 4[34].

In the litterature there exist two[35] different approaches in connection with deciding for an appropriate numerator. First, removal of the market price of risk from the bond-price process - where the money-market account is used as numerator, and where the new process is known as the risk-neutral process. The second approach do not use an instrument with a deterministic price-process as numerator, but instead a T-period zero-coupon bond, for $t < T < \tau$, as numerator.

As mentioned above we will denote the probability measures associated with the use of the money market account as numerator as Q, and we will denote the probability measure associated with the use of the T-period zero-coupon bond as numerator as Q_T.

[34] For an excellent survey of the forward martingale measure, see Musiela and Rutkowski (1997 section 13.2.2).

[35] Here we have disregarded numeraire with are associated with assets denominated in another currency.