Fixed Income Research

The Pricing of Bermudan Swaptions by Simulation

Claus Madsen

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BG Bank
Højbro Plads 10
1015 København K

A Bermudan Swaption (BS)

- A Bermudan Swaption (BS) is an option on a swap that can only be exercised at discrete points in times. Usually these times coincide with the payment dates on the swap $T_F \leq T_L < T$
- Can be defined by 3 dates - where
  - T_F - First strike date (called the lockout period)
  - T_L - Last strike date
  - T - Maturity of the swap

- **Remark: Under these assumptions a Bermudan Swaption (BS) is equal to a Bermudan option on a coupon bond with a strike equal to the par value of the bond**
- **Remark: Another type of Bermudan Swaption (BS) is a Constant Maturity Bermudan Swaption (CMBS)**

Free Boundary problems - I

- We can formulate the free-boundary problem as follows:
- At every time $t \geq T_F$ (for t belonging to the finite set of stopping times τ) up until the final exercise date T_L there will be some critical value P*(t) of the underlying security such that it is optimal to exercise the option if P(t) falls below this critical value. This set of critical values P*(t) forms the early exercise boundary.

- **Remark: Where by nature this optimal exercise boundary is a free boundary - that is the boundary is not given explicitly, but has to be determined as an integral part of the pricing process**

Free Boundary problems - II

- Free boundary problems are handled straightforward in Lattice models because pricing is done by backward-Induction:

$$C_i = \max\left[\phi \cdot (P_{i+1}(t, T-t) - K); \mathrm{d} \cdot C_{i+1}\right]$$

- φ is either -1 or 1. (if equal to -1 => we are considering a payer swaption)

- **Remark: As Monte Carlo works by evolving the underlying state variable/s forward through time it cannot know when it is optimal to exercise - that is it cannot locate the free boundary**

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The Traditional modelling Approach - spot-rate models

- Popular interest models in the Market is:
  - one-factor models - for example Hull and White (HW) (1990), Black and Karasinski (BK) (1991) and Black, Derman and Toy (BDT) (1990) - or some two-factor models like Hull and White (1996) or Ritchken and Sankarasubramanian (1995)

- **Remark: All these models can be implemented numerically in low-dimensional lattices (such as finite differences or binomial trees) - which makes the pricing of american style securities straightforward**

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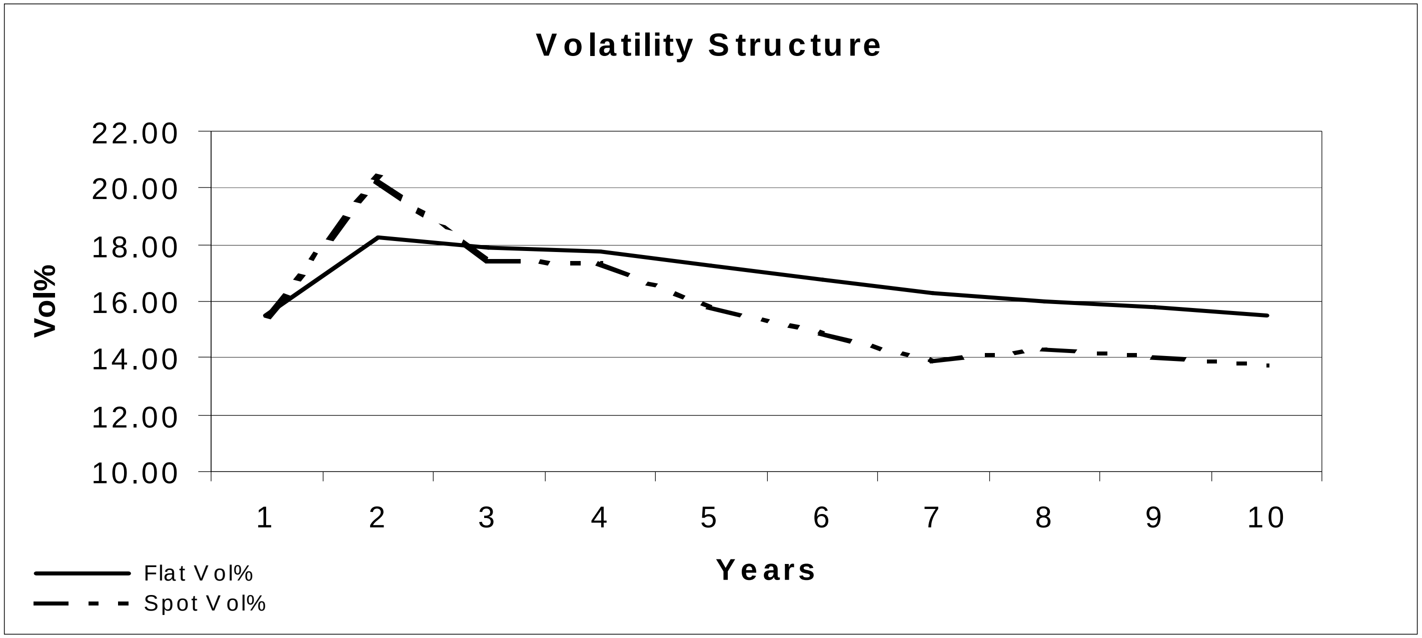
Spot-Rate models and the Volatility Structure - I



- **Flat-Volatility: Sometimes called Black volatility - is the one quoted by brokers. (Assume that each caplet is priced by the same volatility)**
- **Spot-Volatility: The volatility that prices each of the caplets in a Cap**

Spot-Rate models and the Volatility Structure - II

- Implying the volatility parameters from Market Data. Here we have two different approaches:
  - Introduce either a time-dependent mean-reversion or spot-volatility parameter in the diffusion process
    - Pros: Allows for exact fit of the observed Vol.structure
    - Problems: Introduce non-stationarity in the Volatility Structure. (The hump usually encountered in the Vol.structure disappear as times goes)
  - Perform a best-fit of the observed Vol. Structure
    - Pros: The Vol. Structure “remains” constant - stationary
    - Problems: An exact fit is in general not possibly - see next slide
- For spot-rate models I prefer the second approach

The Libor Market Model (LM) - I

- Some stylized facts:
  - It is expressed in terms of discrete time forward rates (as opposed to the HJM with is formulated in continuous time)
  - In the classic form it assume that forward-rates are lognormal
  - Extension to multiple factors are straightforward
- It turns out that closed form solutions for caplets can be derived – which are similar to Black's formula – see slide IV
- It also turns out that an approximate Black formula for swaptions can be derived – see slide IV

- Observed Cap volatility is automatically matched by the LM model

- **Remark: Originally introduced by Brace, Gatarek and Musiela (BGM) (1997) and Miltersen, Sandmann and Sondermann (MSS) (1997) and Jamshidian (1997). Sometimes the Libor Market Model is referred to as the BGM-model.**

The Libor Market Model (LM) - II

- In the LM model the focus is on discretely compounded forward rates, $f(t,T,\delta)$ for the period $[T, T + \delta]$ as seen from time t. $f(t,T,\delta)$ can be expressed as:

$$f(t,T,\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right)$$

- Combining this result with the assumption that the price of a discount bond is governed by the following risk-neutral general SDE:

- $$dP(t,T) = r(t)P(t,T)dt - P(t,T)v(t,T)dW_t$$

- And assuming log-normal diffusion for the forward-rates gives us the process for the forward-rates in the LM model – see next slide

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The Libor Market Model - III

- The process for the forward-rates can be written as:

$$f(t,T,\delta) = f(t,T,\delta)\gamma(t,T)v(t,T)dt + f(t,T,\delta)\gamma(t,T)dW_t$$

- Where the bond-price volatility function – v(t,T) – is defined as:

$$v(t,T) = \sum_{k=1}^{\frac{T-t}{\delta}} \frac{\delta f(t,T,\delta)\gamma(t,T-k\delta)}{1+\delta f(t,T,\delta)}$$

- For the purpose of pricing it is convenient to work under the forward adjusted risk-measure. Under the following transformation the forward-rates f(t,T,δ) becomes martingales under the T+δ forward measure:

- $$W_t^{T+\delta} = W_t + \int_0^t v(s,T+\delta)ds$$

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The Libor Market Model - IV

- Using Black's formula - closed form solutions for caplets exist – where the volatility is defined as:
- $$\sigma^2_{Black} = \frac{1}{T-t}\int_t^T \gamma(s,T)^2\, ds$$

- Again using Black's formula an approximate closed form solution for swaptions can be derived – because of lack of space omitted here - see James and Webber (2000) section 8.3.1 page 210

- **Remark: This approximate swaption formula is quite accurate in practice – see BGM (1997) or Hull (2000)**
- **An even better approximation - the shape-corrector method of Jaeckel and Rebonato (2000)**

Volatility Stationarity - I

- Stationarity means:

$$\gamma(t,T) = \gamma(T-t)$$

- That is – the volatility is only a function of time to maturity

- The stationarity assumption imply one of the two following sentences:
  - Volatilities are identical for all fixed time to maturities
  - Volatilities change over time as the time to maturity changes

- **Remark: The good news is that the observed volatility structure in this case will not change as time goes**
- **Remark: Much work on the BGM model has however been on the non-stationary case, for example Hull and White (2000) and Rebonato (1999). It though seems that recently focus has changed in favor for the stationary models**

Libor Market Models – the last slide

- In generel the LM model can be calibrated to both swaption-and cap-volatilities (see Rebonato (1999), Sidenius (2000) and Brace and Womersley (2000)) even in the case of stationarity – but at the expense of introducing multiple factors. (As an example let me mention that Sidenius (2000) consider 10 factors) - see the presentation “The Libor Market Model - calibration to market prices”.
- However, the flexibility of the LM model does not come for free:
  - The high dimensionality means that pricing has to be done by Monte Carlo simulation
  - This fact leads to 2 problems:
    - **Slow convergence**
    - **How to handle free boundary problems**

- **Remark: The problem of slow convergence can in principle be more or less handled by one or more so-called variance-reduction techniques**

Monte Carlo Simulation of Libor Market Models

- Using the Euler discretization method we can simulate the forward-rates using the following equation:
- $$f(t+\Delta,T)=f(t,T)\exp\left[\gamma(T-t)\cdot\left(\varepsilon\sqrt{\Delta}+\left(v(t,T)-\frac{1}{2}\gamma(T-t)\right)\Delta\right)\right]$$
- As mentioned in Sidenius (2000) we have to recognize that in continuous time v(t,T) does not specify the value for T=t. This turn out to indicate that for the purpose of simulation v(t,T) has to be defined as:

$$v(t,T)=\sum_{k=0}^{T-\frac{t+1}{N}}\frac{\delta f(t,T-k)\gamma(T-k)}{1+\delta f(t,T-k)}$$

- N is the number of time-steps in the simulation and T/N are the length of the time-steps
- **Remark: A good extension is the predictor-corrector method of Jaeckel, Joshi and Hunter (2000) - which models the drift as indirectly stochastic**
-

Bermudan Swaptions in the Libor Market Model

- The question is - Is it possible to price American style securities on a Libor Market Model simulation?
- A lot of information in simulation:
  - We have arbitrage-free samples of a number of yield-curves
  - We have unconditional probability information - that is we can produce unconditional probability expectations
- For most instrument we can calculate the price based solely on the simulation yield-curves
- **However:** To value American style securities we also need the conditional expectations of the pay-off for to be able to figure out if it is optimal to exercise or not

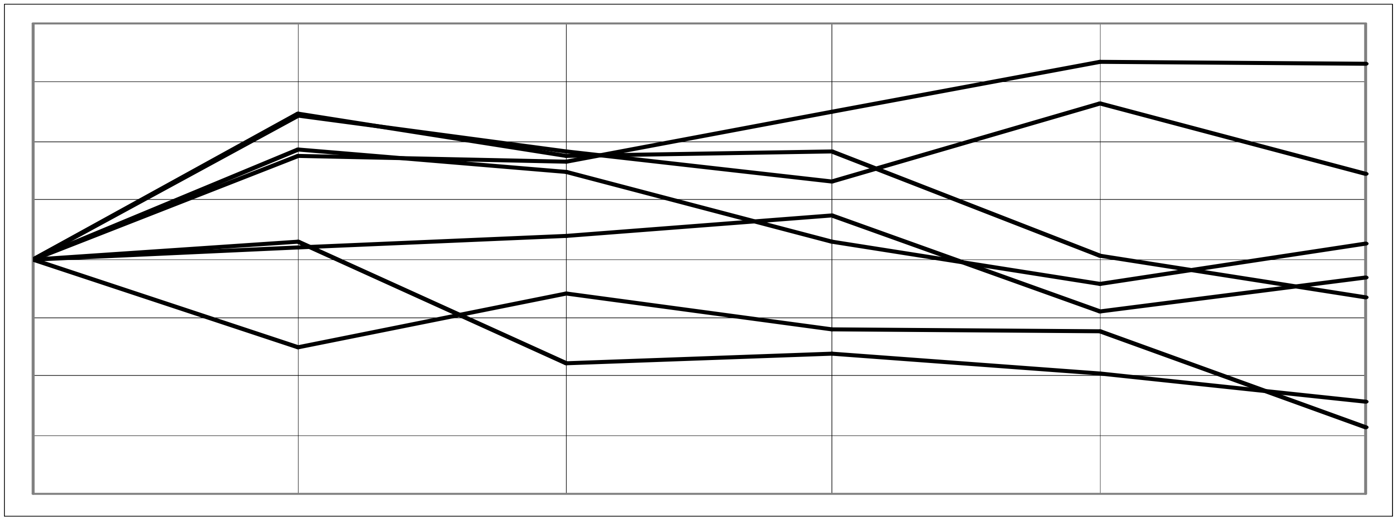
Latticed based approaches

- Non-recombining trees - see for example Gatarek (1996) $\frac{[(m+1)^{N+1}-1]}{m}$
  - Problems: The number of nodes grows exponentially in the number of time steps. For N time-steps and in an m-factor model we have nodes - which makes the method unfeasible

- However it is possible to specify a feasible non-recombining tree method - see later

The Markov Chain Model (MCM) of Carr and Yang (CY)

- Relies on the Stratification method of Barraquand and Martineau (1995)



- They bundle paths together for the money-market account
- The yield-curve for a given state is defined as the average of all the yield-curves that pass through a particulary bucket

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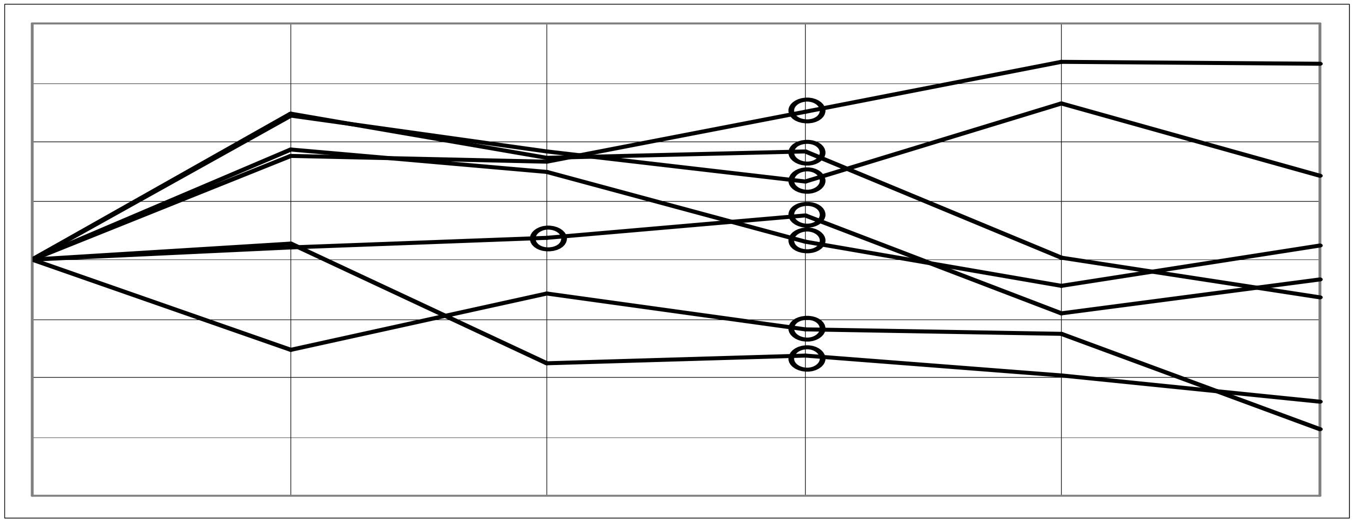
The method of Clewlow and Strickland (CS) (1998)

- As Carr and Yang they also reduce the exercise region to the state of a single variable
- More precisely they use a one-factor Hull and White model implemented in a lattice and obtain the early exercise boundary by determining the critical bond price via backward-Induction
- This boundary are then afterwards used in a full Monte Carlo simulation of multi-factor models (in their case a 2-factor model)

The Stochastic Mesh method from Broadie and Glasserman (BG) (1997)



- The conditional expectation is estimated as the weighted average of the pay-offs one step further on. (Ratio of the conditional density function and the unconditional density function)

- **Remark: Pedersen (1999) suggest using digital caplets in determining the weights.**

Longstaff and Schwartz (LS) and Andersen's boundary optimization technique

- Both models have the following in common: Decision on the early exercise depends on the intrinsic value of the option and the values of still active european options
- Andersen's method works as follows:
  - step 1: Specify the functional form for the exercise strategy: (Good choice is to exercise if C(t) > H(t) (Which is only optimal in a one-factor setting)
  - step 2: Run a MC and store for all times and all paths the intrinsic value + the discount factor
  - step 3: Compute the time-dependent function H(t) such that the value of the Bermudan option is maximised
  - step 4: Generate a new and larger simulation (around 2x) to price the Bermudan option under the exercise strategy in step 3
- Longstaff and Schwartz's is a regression based method with contrary to Andersen focus on simulating the holding value

Jaeckel's method - I

- Jaeckel's method follows structurally along the same lines as Andersen's method. However Jaeckel's method does not require approximative evaluations of option values during the simulation itself

- Instead a parametric exercise decision function is specified as follows:

$$E_i(f(t_i)) = \Phi\left(\left[x_{i1} + \frac{x_{i2}S_{i+1}(0)}{S_{i+1}(t_i) + x_{i2}}\right] - f_i(t_i)\right)$$

- where $\Phi = -1$ for a payer swaption and $\Phi = +1$ for a receiver swaption. (This function is hyperbolic in S_{i+1})

Jaeckel's method - II

- The reason for chosen that functional form for the optimal exercise region was in order to obtain a reasonable separation of exercise and non-exercise regions as much as possible
- In that connection it seems that there are evidence that a projection of the first forward-rate f_i on the swaprate S_{i+1} do manage to do the job quite well

Some observations - I

- In MCM a better approach than bundling the yield-curves (as also pointed out by CY) would be to bundle the payoff space instead of the state space. Overall conclusion seems to indicate that the MCM is better suited for one-factor models
- With BG's Stochastic Mesh method there is the issue of computational time (see for example BG table 4).
  - Results from Pedersen (1999) however indicate that it is computationally much better to use digital caplets - (for a 50/500 mesh - 50 times faster)
- With respect to the CS method it can be argued that the optimal exercise region derived from a one-factor setting has limited use in a multi-factor framework (at least in some cases)

Some observations - II

- Andersen’s method is in principle related to the CS method - however here the optimal exercise boundary is determined in the same model that are used for pricing
- LS’s method differ mainly from Andersen’s method in the sense that step 4 is omitted here. That is we use the same simulated paths to determine the value of the security in question that we use to compute the optimal exercise decision

- **Remark: Andersen’s decide to separate the simulations in order to prevent what he calles “perfect foresigth biases” - see footnote 9 page 14.**
- **In a sense the LS method is similar to the stochastic mesh method - but with regression weights rather than likelihood ration weights**

Some observations - III

- Jaeckel's method differ only from Andersen's method because of the following 2 things:
  - No approximative option values are used in estimating the optimal exercise boundary
  - The exercise decision is designed in order to able to have a reasonable separation of exercise and non-exercise regions
- In general it is concluded that non-Recombining trees are not recommendable - however as we will see shortly it is feasible (see Jaeckel 2000))

MC and the optimal exercise boundary

- When we wish to price options of American style - then we really need to compare the expected payoff as seen from any node with the intrinsic value. This entails that the only method that can give an unbiased result is a non-recombining tree.

- Recently Broadie (2001) has come up with a method to determine the lower bound of the american style securities valued using Monte Carlo simulation

- **Remark: Andersen and Broadie is currently working on how to asses the upper bound of Bermudan swaptions in the Libor Market Model - but no work has been published so far**

A non-recombining tree approach - I

- A feasible non-recombining tree method for the Libor Market Model:
- Lets assume we have a matrix $M \in \Re^{d,m}$ whose rows consist of the vectors ε to be used for each realisation of the evolved yield-curve as given by slide 14 (We have an m-factor model)
- If we wish to assign equal probability to each of the d realisations it turns out that the elements of the matrix M describe the Cartesian coordinates of a perfect simplex (all the angles is equal) in m dimensions
- The smallest d for which it is possible to construct the M-matrix so it satisfies the above requirements is m + 1 – that is we need a minimum of m + 1 branches out of each node

A non-recombining tree approach - II

- Cartesian Coordinate system:

$$c_{i,j} = \begin{cases} -\sqrt{\dfrac{m+1}{j(j+1)}} & \text{for } j \geq i \\ \sqrt{\dfrac{j(m+1)}{j+1}} & \text{for } j = i-1 \\ 0 & \text{for } j < i-1 \end{cases}$$

- Example matrix for m = 5

-1.7321	-1	-0.7071	-0.5477	-0.4472
1.7321	-1	-0.7071	-0.5477	-0.4472
0	2	-0.7071	-0.5477	-0.4472
0	0	2.1213	-0.5477	-0.4472
0	0	0	2.1909	-0.4472
0	0	0	0	2.2361

- **Remark: For m = 1 we get the M-matrix to be: $c_{1,1}$ = -1, $c_{2,1}$ = 1**
- **Remark: Extending to more branches than m + 1 is straightforward**

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A non-recombining tree approach - III

- However, in order to benefit for the adding of branches we might want to spread them as evenly as possible. One natural candidate is a matrix were each column are symmetric around 0 (zero) – this can be thought of as a suitable alignment of the simplex

- The result for the matrix from the last slide is:

- **Remark: Further improvement can be achived by combining the simulation procedure with a technique called Alternative Simplex Direction (ASD) - which entails switching the signs of all the simplex coordinates in every step**

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Some Preliminary Conclusions - I

- In generel the results reported looks promising but as mentioned by James and Webber (2000) - section 13.1.5 page 353
  - *"..on the whole these methods are computationally highly intensive, and allthough promising their value is yet to be proved"*

- Another interesting question is the factor dependency for Bermudan swaption where no clear and general accepted result so far has appeared. The issue here is: is the price of a Bermudan swaption sensitive to the number of factors?

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Some Preliminary Conclusions - II

- A last interesting observation is: As most of the value in an American swaption lies in the exercise opportunity immediately prior to reset (payment dates on the swap) - we can approximate the value of an American swaption by a Bermudan swaption that can be exercised at the time step immediately prior to reset

- **Remark: We are in the process of performing a more detailed analyses of the practical use both in terms of computational time and results for some of the approaches mentioned here. We expect that an updated slide-show will be available around december 2001. Anyone interested in receiving a copy can e-mail me at:**

  claus_a_madsen@yahoo.com