Inflation-Linked Products

20. February 2007

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Agenda

The Market - I

- Inflation Swaps (USD, EUR, GBP and JPY)
  - Zero-Coupon Swap
  - YoY (Year-on-Year)

- Index-Linkers (USD, GBP, FRF, SEK, CAD)
  - CIB (IZCB)
  - (IIB – Australia 1988)
  - (CPB – Turkey 1997-1999)
  - (IAB – Australia)

The Market - II

- Zero-Coupon Inflation-Swaps:
  - Standard broker market swaps
  - Payer pays at maturity: $$N\left(\frac{HICP(m-l)}{HICP(s-l)}-1\right)$$
  - Receiver recieves at maturity: $$N\left[(1+K)^T-1\right]$$
- where s = Start-Date, m = Maturity-Date, l = Lag (typically 2 or 3 month), N is the principal and T = number of years.

- YoY Inflation-Swaps:

- Payer pays at maturity: $$ND\left(\frac{HICP(p-l)}{HICP(p-(l+12))}-1\right)$$

- Receiver receives at maturity: NDK. (D is the day-count fraction)

The Market - III

- The Indexing Process for Index-Linkers uses as standard the so-called Canadian-Model, which can be explained as follows:

- The index-ratio for a given settlement date is calculated as follows:

$$IR_{SettlementDate} = \frac{CPI_{ref}}{CPI_{base}}$$

- The reference CPI for the first day of any calendar month is the CPI for the calendar month falling 3 month earlier, that is the reference CPI for 1 june corresponds to the CPI for march etc.

- The reference day for any other day in the month is calculated by linear interpolation – see the next slide..

The Market - IV

- The formula that is used to calculate the reference CPI can be written as:

$$CPI_{ref} = CPI_{ref}^{m} + \left(\frac{t-1}{d}\right)\left[CPI_{ref}^{m+1} - CPI_{ref}^{m}\right]$$

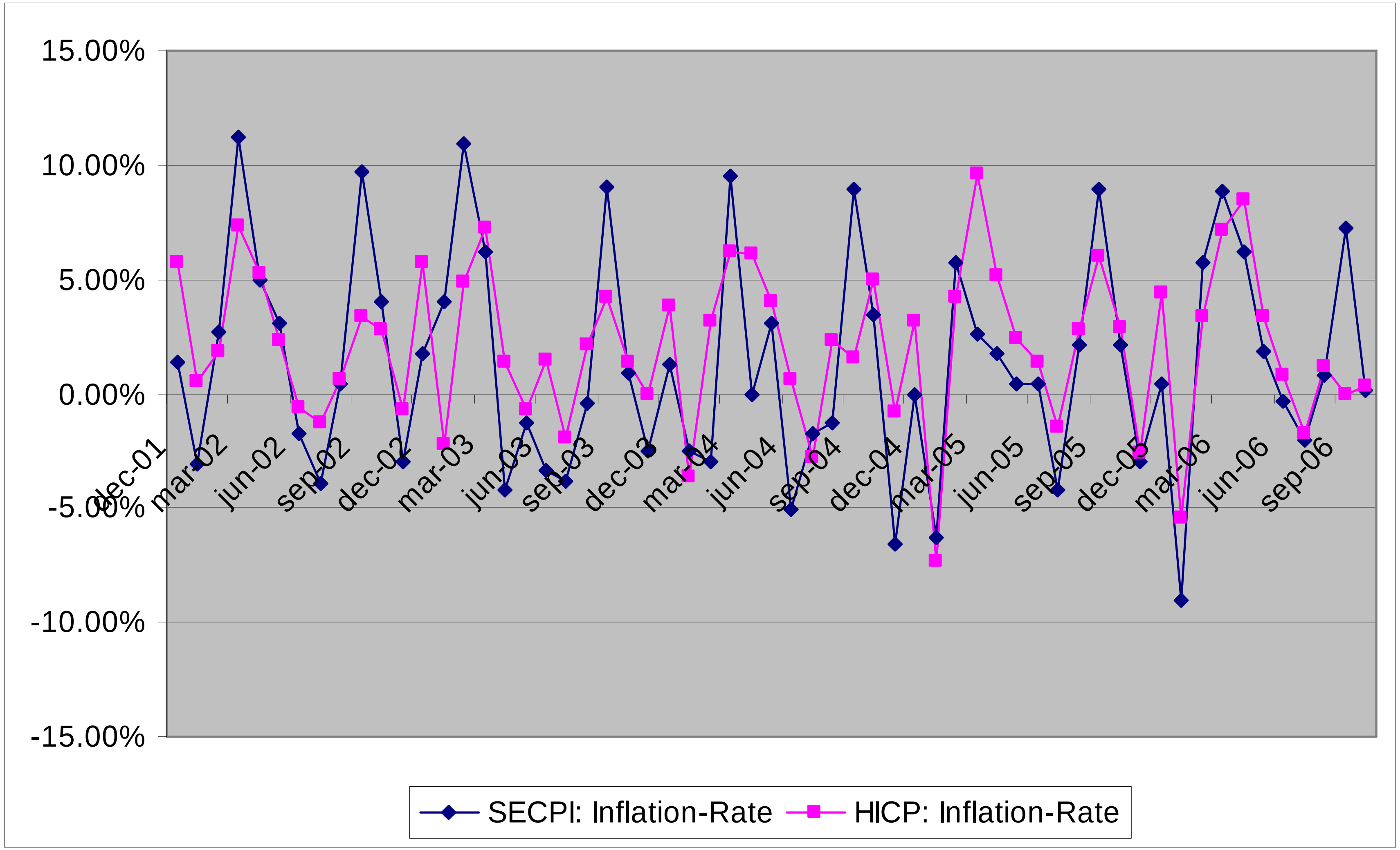
- Where d = the number of days in the calendar month in which the settlement date falls, t = the calendar day corresponding to the settlement date (for Swedish linkers if = 31 then t = 30). Reference CPI(m) is the reference CPI for the first day of the calendar month in which the settlement date falls and reference CPI(m+1) is the reference CPI for the first day of the calendar month immediately following the settlement date.

- **Remarks:** In many cases there is a deflator-floor at 0% (i.e. all the OATs and most of the Swedish (after 1999))

Agenda

The CPI-Curve and Seasonality - I

• Example of known CPI-Data (shown as a yearly inflation-rate):



The CPI-Curve and Seasonality - II

- Due to seasonality in the underlying price indices, the carry for linkers is much larger than and more volatile than for nominal bonds
- Since forwards depend on future price indices, for Index-Linkers with a 3M indexation lag "arbitrage-free" forward valuations can only be calculated about 45 days ahead
- For longer horizons inflation forecasts can be used to calculate linkers carry outright versus nominal bonds (BE-protection)

- Definitions:
- Real-Carry in bp = forward real yield – spot real yield
- Forward BE = forward nominal yield – forward real yield
- BE-protection = spot BE – forward BE (corresponds to the effective carry embedded in a long BE position)

The CPI-Curve and Seasonality – Carry Calculation - I

- Consider the SEK3106 (1% 2012) trading at a real clean price of 96.628 as of 9. November 2006. Real accrued IR is 0.6194 and Base CPI = 280.4. The 3m repo rate is 2%.

- Real Spot Yield = 1.66%
- Spot Nominal Dirty Price = (96.628 + 0.6194)x(285.09933/280.4) = 98.8772
- 3M forward Nominal Dirty Price = 98.8772 x (1+0.02 x 92/360) = 99.3826
- 3M forward real Dirty Price = 99.3826 x (280.4/286) = 97.4366 (assuming CPI = 286)
- 3M forward real Yield = 1.70%
- 3M Carry in bp = 1.70% - 1.66% = 4bp

The CPI-Curve and Seasonality – Carry Calculation - II

- We use the SEK1046 (5.5% 2012) to perform the B/E protection calculation. The SEK1046 is trading at a clean price of 109.729

- Nominal Spot Yield = 3.63%
- 3M forward Dirty Price = (109.729 + 0.596) x (1+0.02 x 92/360) = 110.889
- 3M forward Nominal Yield = 3.65%
- Spot Breakeven = 3.63% - 1.66% = 1.97%
- 3M forward Breakeven = 3.65% - 1.70% = 1.95%
- 3M B/E protection in bp = 1.97% -1.95% = 2bp

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Building the CPI-Curve - I

- The CPI-Curve is the fundamental building block for inflation products:

- **Example 1:**

- Find the constant inflation-rate K:
  - Derive the CF on the floating-leg using the forward CPI-Curve
  - Calculate the PV of the floating-leg
  - Imply the inflation-rate using the previous calculated PV
    - - That is: Same procedure as for Interest-Rate Swaps

- **Example 2:**

- Compare Index Linkers with nominal Bonds:
  - Derive the nominal CF for Index-Linkers
  - Price using the nominal YC
  - Etc....

Building the CPI-Curve - II

- The CPI-Curve can be estimated using either Index-Linkers or ZC Inflation-Swaps (or other liquid instruments!)

- In general we recommend that seasonality is taken into account when estimating the CPI-Curve - this can for example be done by using CPI information for the previous 12 months

- Let's look at bit more on YC Inflation-Swaps before proceeding with some examples…

- From the foreign-currency analog we have that the price of ZCIS can be written as:

$$ZCIS(t,T,I_0,N) = N\left[\frac{I(t)}{I_0}P_r(t,T) - P_n(t,T)\right]$$

- which for t = 0 simplifies to:

$$ZCIS(0,T,N) = N\left[P_r(0,T) - P_n(0,T)\right]$$

Building the CPI-Curve - III

- The market quotes the values K = K(T) for some maturities T. This allows us to express the discount factor for maturity T in the real economy as:

- $$P_r(0,T) = P_n(0,T)[1+K(T)]^T$$

- Kazziha (1999) defines the forward CPI at time t as the fixed amount X that is to be exchanged at time T for CPI I(T), for which a swap has zero value at time t – that is (this is actually consistent with the foreign-currency analogy):

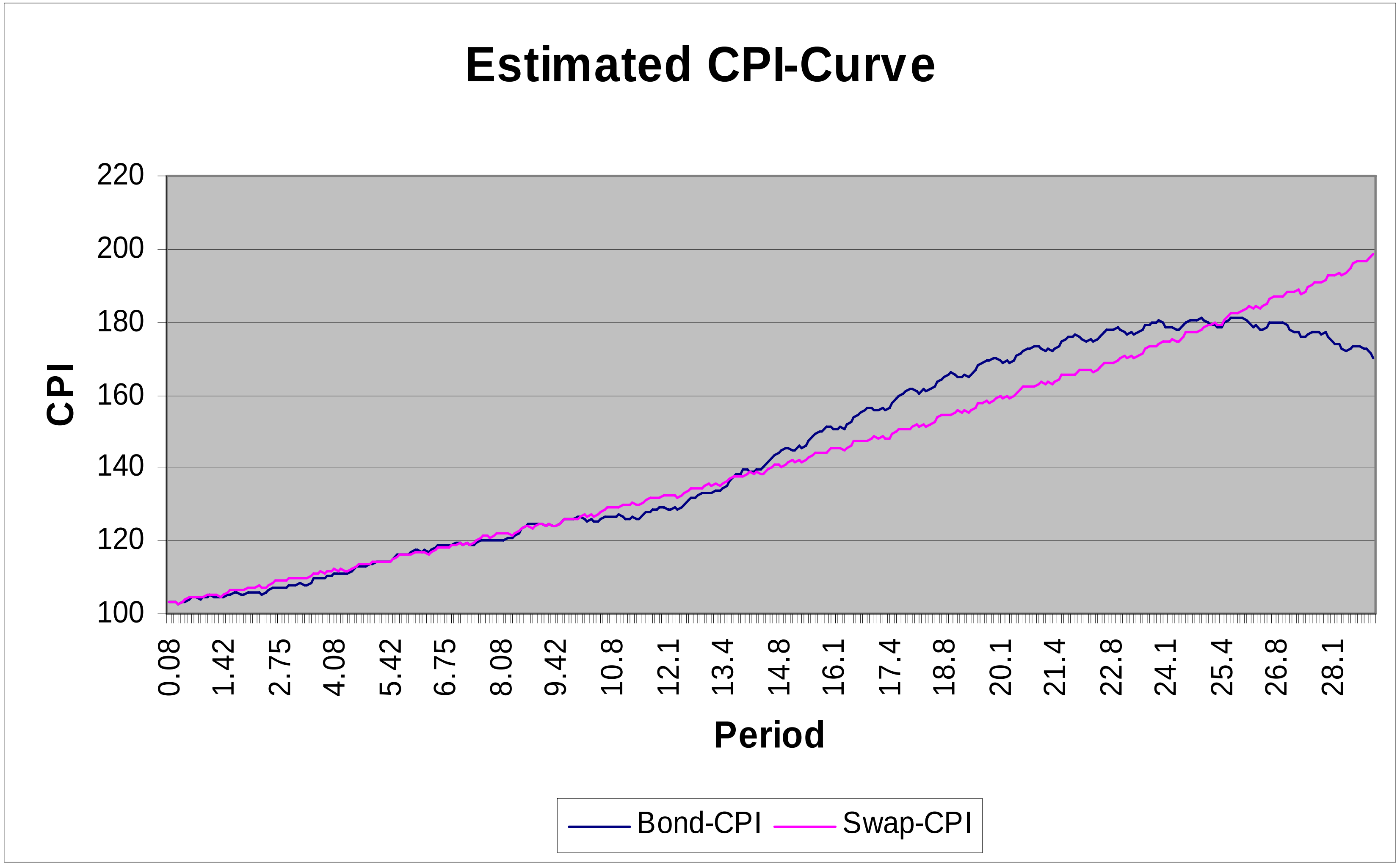
$$I(t)P_r(t,T) = XP_n(t,T)$$

- Combining the 2 equations yields the main result:

$$I_T(0) = I(0)[1+K(T)]^T$$

- Which states that the pricing system is only based on forward CPIs and nominal rates – no foreign-currency analogy is needed

Building the CPI-Curve - IV



Building the CPI-Curve – V (ZCIS Data)

FINE ANALYTICS

	ZCIS
1	1.94%
2	1.97%
3	2.10%
4	2.09%
5	2.06%
6	2.11%
7	2.08%
8	2.12%
9	2.12%
10	2.09%
12	2.12%
15	2.13%
20	2.19%
25	2.22%
30	2.28%

Building the CPI-Curve – VI (Index-Linkers Data)

FINE ANALYTICS

XBondID	Clean Real Price	Real Accrued IR	Dirty Real Price	Index-Factor	Dirty Nominal Price	Nominal Fair Price	Maturity
IT0003532915	100.52	0.78	101.30	1.06583	107.97	107.97	20080915
IT0003805998	97.78	0.45	98.23	1.04401	102.56	102.56	20100915
FR0000188013	108.68	0.35	109.04	1.10123	120.07	120.07	20120725
IT0003625909	103.22	1.02	104.25	1.06583	111.11	111.11	20140915
FR0010135525	100.36	0.19	100.55	1.04427	105.00	105.00	20150725
DE0001030500	98.70	0.59	99.29	1.01500	100.78	100.78	20160415
IT0004085210	102.22	1.00	103.22	1.01500	104.77	104.77	20170915
FR0010050559	107.05	0.27	107.32	1.06568	114.36	114.36	20200725
FR0000188799	130.50	0.37	130.88	1.07975	141.31	141.31	20320725
IT0003745541	107.11	1.12	108.22	1.04401	112.99	112.99	20350915

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Inflation-and Interest-Rate Risk - I

- The price of an Index Linker can be expressed as follows:

$$P_n = \sum_{i=1}^{m} \frac{C_R \left[1 + \frac{I(m)}{I(0)} \right]}{(1+r)^m \left[1 + \frac{I(m)}{I(0)} \right]}$$

- In the market the nominal price is being calculated as (this assumes cancelling of inflation dependencies):

$$P_n = (1+i_0) \sum_{i=1}^{m} \frac{C_R}{(1+r)^m}$$

Inflation-and Interest-Rate Risk - II

- In general the modified duration of an Index Linker is calculated as follows:

$$MD = -\frac{1}{P}\frac{dP}{dr}$$

- Calculating MD this way gives rise to ”huge” MD numbers due to the fact the real-rate in general is ”small” – in a real-rate setting this concept however has some use. However, care must be taken when comparing with nominal bonds.

- The reasoning is clear from the Fisher equation: (1+y) = (1+r)(1+i) – disregarding risk premium. A nominal bond has sensitivity both to real-rates and to the expected inflation, which means that its duration measures the bonds sensitivity to some combination of both these factors – this is in hedging referred to as double duration.

Inflation-and Interest-Rate Risk - III

- In order to be able to compare Index Linkers with nominal bonds it has become common practise to use the beta measure.

- Beta is defined as the yield sensitivity of an Index Linker to a change in the nominal yield. Given that real-rates are generally assumed to be less volatile than nominal yields, beta will usually be less than one (numbers between 0.6-0.8 are fairly common)

- In principle a beta estimate will allow an investor to translate real duration into nominal duration – however, beta is not stable!

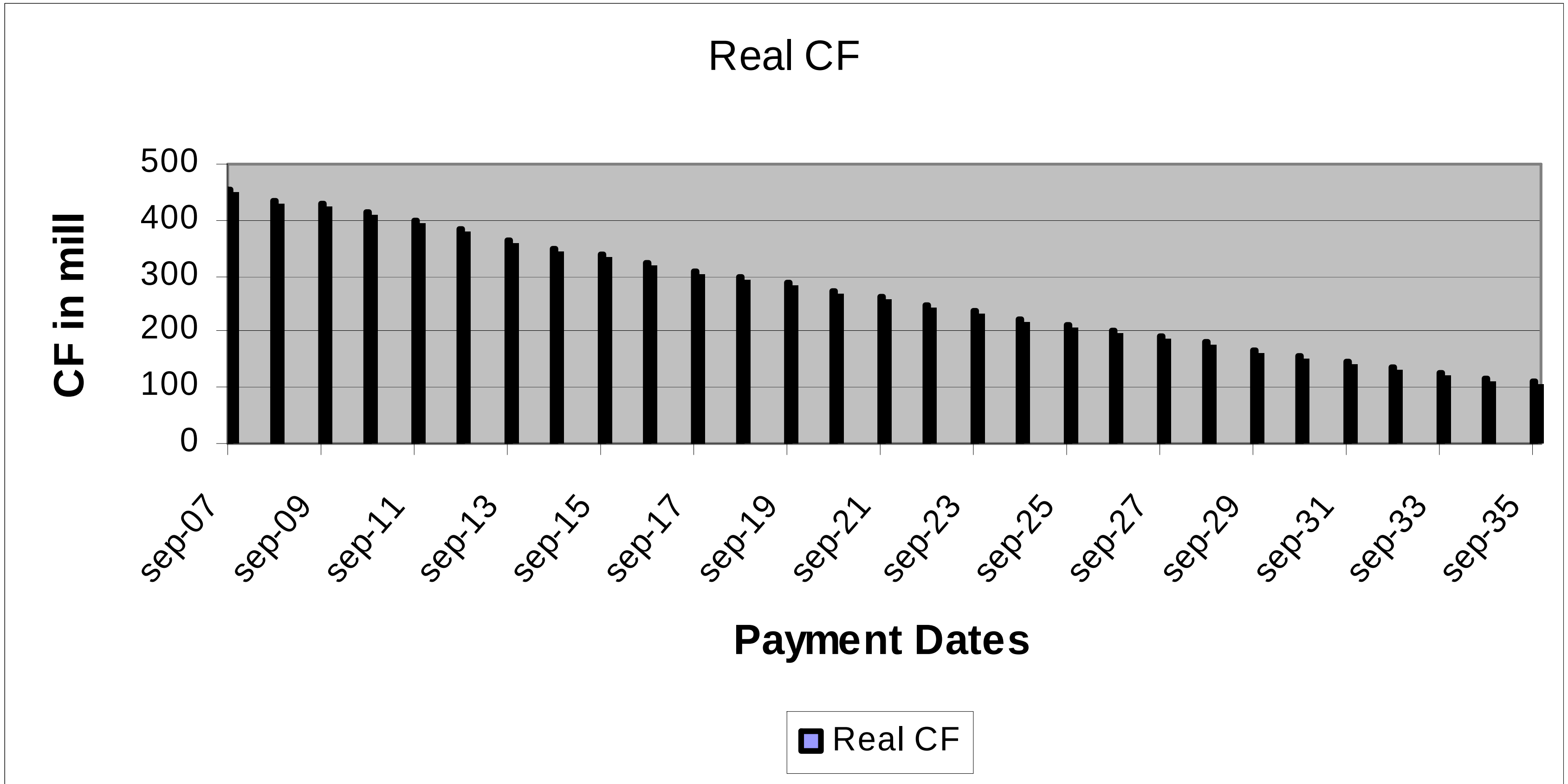
- Another approach is as follows (which could be termed the Direct Approach):
  - Derive the forward CPI-Curve
  - Estimate the CF using the forward CPI-Curve
  - Calculate sensitivities using the nominal yield-curve

Agenda

FINE ANALYTICS

Liability Management - I

•Let us consider the following case. We wish to hedge the combined inflation/interest rate exposure on the following real liability CF (in SEK):



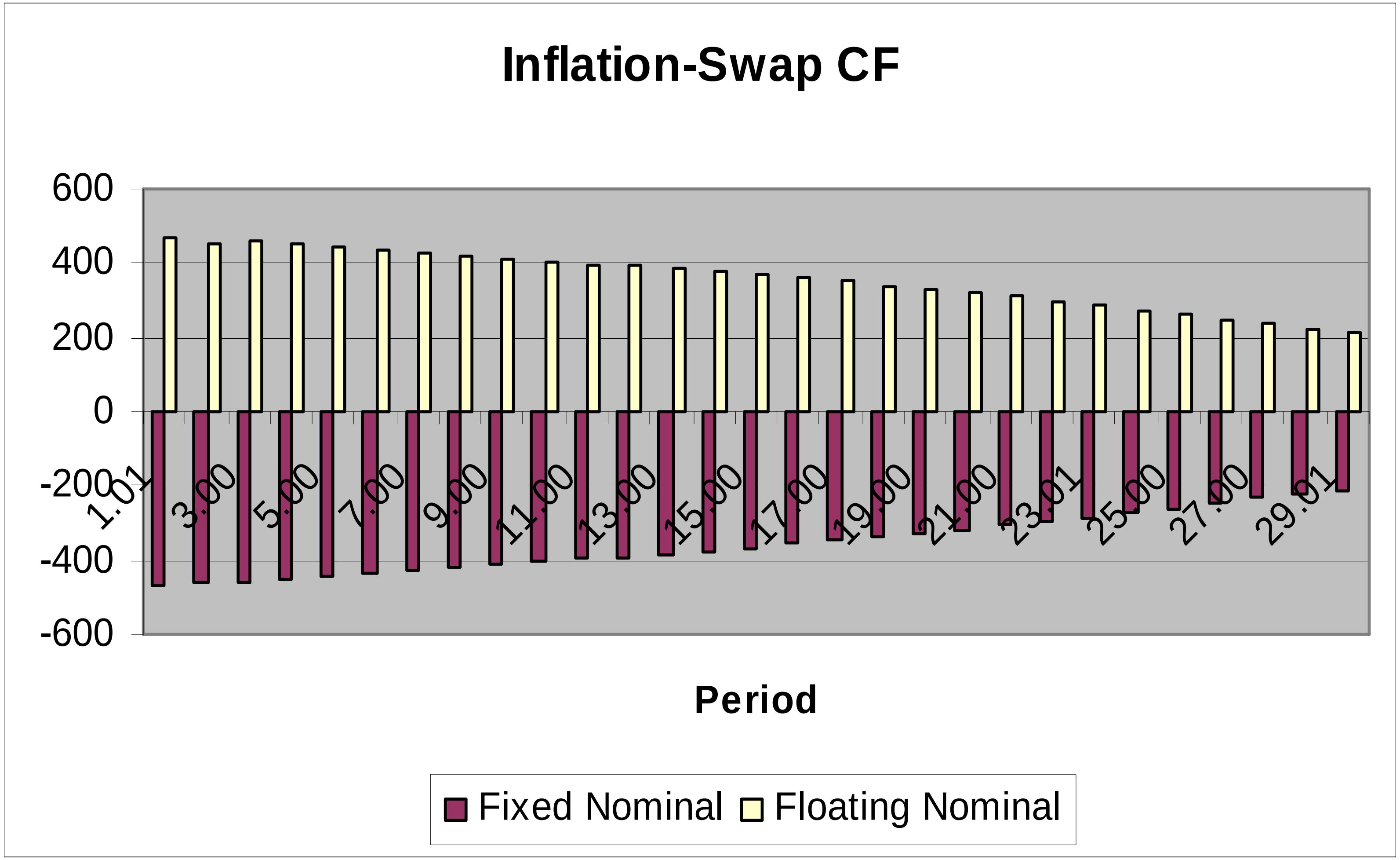
Liability Management - II

- We could use Index Linkers – but in that case a perfect hedge is not attainable, there exist only 6 Linkers in Sweden – limited available structures.
- By using Inflation-Swaps it is possible to tailor make the CF structure

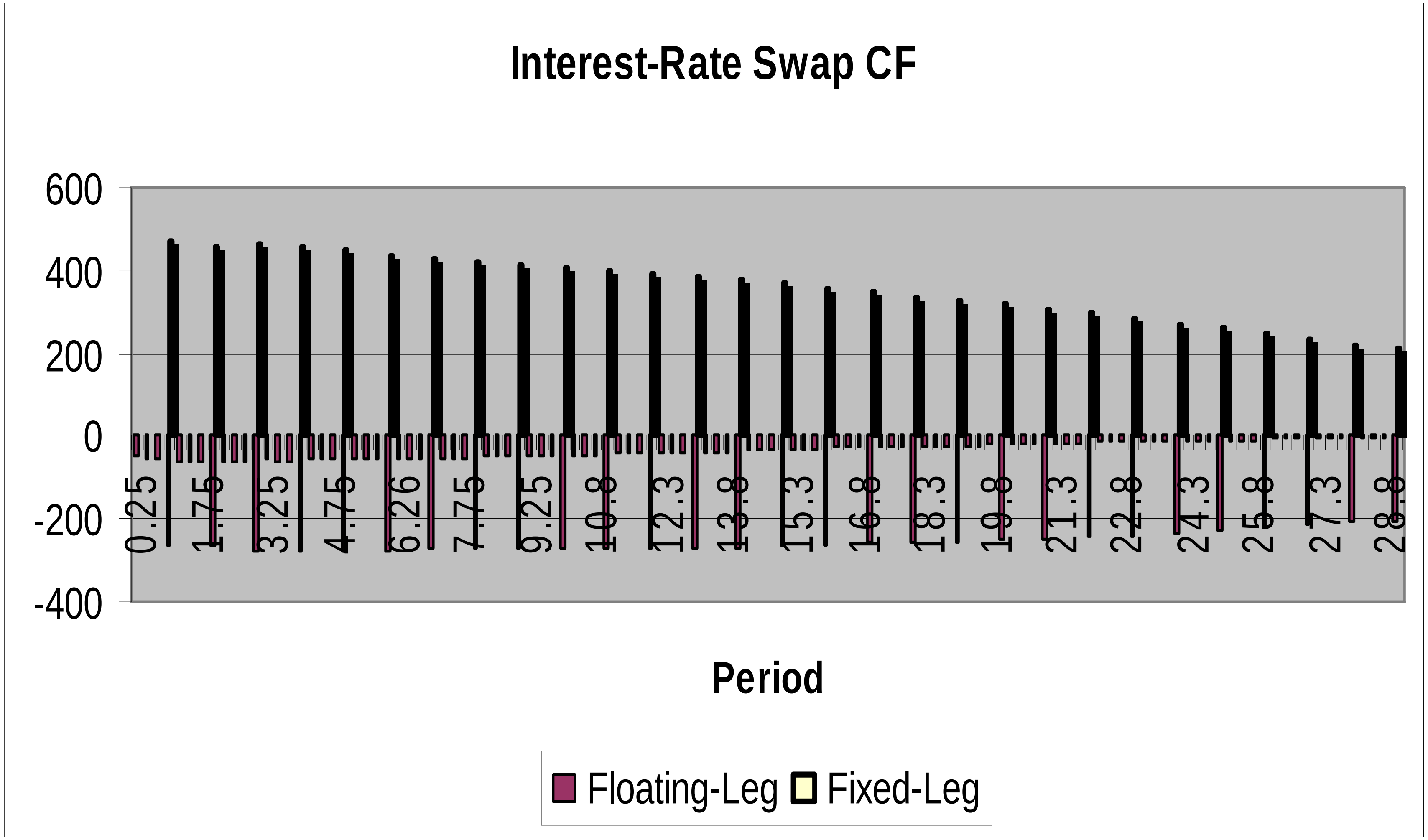
- We need to do the following trades:
- Inflation-Swap: Recieve floating – pay fixed
- Interest-Rate Swap: Receive fixed – pay floating

- End Result: We will have translated our long term inflation risk and interest rate risk into a short term interest rate risk. (Assuming that the Inflation-Swap exactly offset the Inflation-Risk on the liabilities!)
- More details on the next slides....

Liability Management - III



Liability Management - IV



Liability Management - V

- So what did we do:
- 1: Using the swedish swap-curve and the forward CPI-Curve we calculated the nominal price of the floating leg of the customized Inflation-Swap
- 2: Given the price of the floating leg of the Inflation-Swap we calculated the Fixed Inflation-Swap Rate
- 3: Given the nominal CF of the fixed leg on the Inflation-Swap we rolled-up the principal pattern for the amortizing Interest-Rate Swap given the fixed Swap-Rate. The roll-up procedure ensures that the nominal CF of the fixed-leg of the Interest-Rate Swap is identical to the nominal CF of the fixed-leg of the Inflation-Swap
  - Important: It is not possible to derive the principal pattern for the Interest-Rate Swap without knowing the Swap-Rate. This means that the Swap-Rate has to be derived without having knowledge to the present value of the floating-leg of the Interest-Rate Swap. One obvious way is to use the nominel yield on the nominel CF on the fixed-leg of the Inflation-Swap as the Swap-Rate – however, other methods is applicable

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Modelling Inflation - I

•Compared to ZCIS the valuation of YYIS (YoY Inflation Swaps) is more involved, as we will se below:

$$YYIS(t,T_{i-1},T_i,D_i,N) = ND_iE_n\left[e^{-\int_t^{T_i} y(u)du}\left(\frac{I(T_i)}{I(T_{i-1})}-1\right)\Big|F_t\right]$$

$$= ND_iE_n\left[e^{-\int_t^{T_{i-1}} y(u)du}\left[e^{-\int_{T_{i-1}}^{T_i} y(u)du}\left(\frac{I(T_i)}{I(T_{i-1})}-1\right)\Big|F_{T_{i-1}}\right]\Big|F_t\right]$$

•It might not be obvious but the inner expectation is a ZCIS(T_{i-1},T_i,I(T $_{i-1}$),1), so we get:

$$ND_iE_n\left[e^{-\int_t^{T_{i-1}} y(u)du}[P_r(T_{i-1},T_i)-P_n(T_{i-1},T)]\Big|F_t\right]$$

$$= ND_iE_n\left[e^{-\int_t^{T_{i-1}} y(u)du}P_r(T_{i-1},T_i)\Big|F_t\right]-ND_iP_n(t,T_i)$$

Modelling Inflation - II

- The last expectation can be viewed as a the nominal price of a derivative that pays in nominal units the real ZC bond price $P_r(T_{i-1}, T_i)$ **at time** T_{i-1}

- In case real-rates where deterministic, then we could write the expectation as:

$$P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})}$$

- However, real rates are stochastic which makes the expectation model dependent.

Modelling Inflation - III

- One candiate model is the Jarrow and Yildrim (2003) model, which can be written as follows:

$$df_n(t,T) = \kappa_n(t,T)dt + v_n(t,T)dW_n^P(t)$$

$$df_r(t,T) = \kappa_r(t,T)dt + v_r(t,T)dW_r^P(t)$$

$$dI(t) = I(t)\mu(t)dt + \sigma_I I(T)dW_I^P(t)$$

- where:

(W_n^P, W_r^P, W_I^P) **is a brownian motion with correlations** $\rho_{\mathbf{n,r}}, \rho_{n,I,}\rho_{r,I}$;

κ_n, κ_r, μ **are adapted processes;**

$v_{\mathbf{n}}, v_r$ **are deterministic functions;**

$\sigma_{\mathbf{I}}$ **is a positive constant**

- To ease calculation we assume that the forward volatilites are affine, more precisely we assume:

$$v_x(t,T) = \sigma_x e^{-\kappa_x(T-t)}$$

Modelling Inflation - IV

- Now we would proceed to rephrase the dynamic in terms of instantaneous short rates under the risk neutral probability measure Q(n). Due to lack of space (and probably time) we will just explain the result:

- It turns our that both nominal rates and real (instantaneous) rates are normally distributed under their respective risk-neutral measures and that the real rate is still an Ornstein-Uhlenbeck process under the nominal measure Q(n). Last we have that the inflation index I(t), at each time t, is lognormally distributed under the measure Q(n).

- The equations are messy but given these assumptions we are actually able to get closed form solutions for YYIS (and for options on inflation) – se next slide....

Modelling Inflation - V

- Lets denote Q(T,n) the nominal T-forward measure for the maturity T. We can then express the value of a YYIS as:

$$YYIS(t,T_{i-1},T_i,D_i,N) = ND_i P_n(t,T_{i-1}) E_n^{T_{i-1}}\left[P_r(T_{i-1},T_i)\middle|F_t\right] - ND_i P_n(t,T_i)$$

- After some tedious calculations it can be shown that the value of a YYIS can be written as:

$$YYIS(t,T_{i-1},T_i,D_i,N) = ND_i P_n(t,T_{i-1}) \frac{P_r(t,T_i)}{P_r(t,T_{i-1})} e^{C(t,T_{i-1},T_i)} - ND_i P_n(t,T_i)$$

for

$$C(t,T_{i-1},T_i) = \sigma_{P_r}(T_{i-1},T_i)\left[\frac{\sigma_{P_r}(t,T_{i-1})}{\sigma_r}\left[\rho_{r,I}\sigma_I - \frac{1}{2}\sigma_{P_r}(t,T_{i-1}) + \frac{\rho_{n,r}\sigma_n}{k_n-k_r}\left[1+\frac{k_r\sigma_{P_n}(t,T_{i-1})}{\sigma_n}\right]\right] - \frac{\rho_{n,r}\sigma_n}{k_n-k_r}\frac{\sigma_{P_n}(t,T_{i-1})}{\sigma_n}\right]$$

and

$$\sigma_{P_x}(t,T) = \frac{\sigma_x}{k_x}\left[1-e^{-k_x(T-t)}\right]$$

Modelling Inflation - VI

- From this it can be seen that the convexity adjustment depends on the instantaneous volatilities of the nominal rate, the real rate and the CPI, on the instantaneous correlation between the nominal and real rates and on the instantaneous correlation between the real rate and th CPI.

- It is also obvious that in the case of deterministic real rates this convexity term disappears. The deterministic case is obtained for: $\sigma_r = 0$

- The advantage of using a Gaussian model for the nominal and real rates is the availability of tractical analytical formulas. However, the possibility of negative rates and the difficulty in obtaining parameter estimates has led to other interesting solutions – see Belgrade, Benhamou and Koechler (2004)

Modelling Inflation - VII

- This approach is the market approach!

- If we use the definition of forward CPI and the fact that (see slide 14) I_T is a martingale under Q_n^T we can also write the value of an YYIS as:

$$YYIS(t,T_{i-1},T_i,D_i,N) = ND_iP(t,T_i)E_n^{T_i}\left[\left(\frac{I(T_i)}{I(T_{i-1})}-1\right)\Big|F_t\right]$$

$$= ND_iP(t,T_i)E_n^{T_i}\left[\left(\frac{I_{T_i}(T_{i-1})}{I_{T_{i-1}}(T_{i-1})}-1\right)\Big|F_t\right]$$

- Before proceeding let's state the following:

- In the market approach it is assumed that both the nominal rates and the real rates follow a Libor Market Model (BGM Model)

Modelling Inflation - VIII

- It turns out that we (under some simplified assumptions) can solve this expectation as follows:

$$E_n^{T_i}\left[\frac{I_{T_i}(T_{i-1})}{I_{T_{i-1}}(T_{i-1})}\middle|F_t\right]=\frac{I_{T_i}(t)}{I_{T_{i-1}}(t)}e^{C(t,T_{i-1},T_i)}$$

for

$$C(t,T_{i-1},T_i)=\sigma_I(T_{i-1})\left[\frac{\rho_{I,n}\sigma_n(T_i)(T_i-T_{i-1})F_n(t,T_{i-1},T_i)}{1+(T_i-T_{i-1})F_n(t,T_{i-1},T_i)}-\rho_{I,i}\sigma_I(T_i)+\sigma_I(T_{i-1})\right](T_{i-1}-t)$$

- Where F(t,x,y) is the simple compounded rate at time t for the expiry maturity pair x,y), $\rho_{I,n}$ is the instantaneous correlation between I(.) and F_n (.,T_{i-1} ,T_i) and

$$\rho_{I,i}dt=dW_{T_{i-1}}^I(t)dW_{T_i}^I(t)$$

- This lead us to the following expression for an YYIS –see next slide:

Modelling Inflation - IX

•

$$YYIS(t,T_{i-1},T_i,D_i,N) = ND_i P_n(t,T_i)\left[\frac{P_r(t,T_i)P_n(t,T_{i-1})}{P_r(t,T_{i-1})P_n(t,T_i)}e^{C(t,T_{i-1},T_i)} - 1\right]$$

• Where the convexity adjustment is defined on the previous slide.

• From this we see that the value of an YYIS depends on the instantaneous volatilities of forward inflation indices and their correlation, the instantanous volatilities of nominal forward rates and the instantaneous correlations between the forward inflation indices and the nominal forward rates.

• This equation can be compared to the expression on slide 33 and here we see that the market model approach does not depend on the volatility of the real rates - this is clearly an advantage. However the drawback is that the formula is based on an approximation which might be too rough for long maturities T_i. It might be worth saying that the formula is exact for $\rho_{I,n}$ set equal to zero.

Modelling Inflation - X

- We could now continue to the pricing of options on inflation and the pricing of more complex options or proceed to another very interesting area – namely parameter estimation!

- In order to try to keep the time-schedule – which we might have passed allready - I will however leave it here – the rest is for another time and another place

- Thank you

- Claus Madsen/20. February 2007