TERM-STRUCTURE DYNAMICS AND THE DETERMINATION OF STATE-VARIABLES - A MULTI-FACTOR APPROACH

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Abstract: In this paper we extend the traditional approach in the literature that uses Principal Component Analysis (PCA) when analyzing the dynamic in the yield-curve by taking into account the dynamic part of the process.

Using a Gaussian framework we establish the relationship between the statistical and dynamic aspects of the linear factor structure taking a probability-based framework as our starting point. In this connection we show that the identification of the state variables gives rise to a Markovian stochastic system. This Markovian structure is set up by first assuming that the dynamics in the state variables follow a multi-factor extended Vasicek model and second that the dynamics in the forward rates is defined in an m-factor model in the HJM framework.

Relying on this approach we suggest a dynamic factor model which is estimated on the Danish market for the period 2 January 1990 - 30 June 1998 using the linear Kalmar filter technique.

We find that all the parameters are significant, and furthermore by performing a multiple regression of the state-variables on the actual interest rate evolution (in sample) this results in a R^2 of no less than 98% for all the 9 maturities we analysed.

As is generally reported in the literature we also find that the level-factor is close to being non-stationary. Furthermore we also find that the two (2) dominant factors are the level-factor and the slope-factor - which are in line with the result from among others Litterman and Scheinkman (1988).

Keywords: Multi-factor models, HJM, Markovian volatility structures, generalized Vasicek, Gaussian models, Kalman filter, Dynamic factor structure

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Term-Structure Dynamics and the Determination of state-variables - a Multi-Factor Approach[1]

1: Introduction

In a number of empirical studies of the dynamics of the yield-curve, a general conclusion is that 3 factors are necessary (and sufficient) for describing the dynamics of the yield-curve, see among others Litterman and Scheinkman (1988).

These 3 factors are further identified as a level factor, a slope factor and a curvature factor.

These studies show that the estimated factor-loadings are of both practical and theoretical relevance as they appear for (as yet) inexplicable reasons to be remarkably stable over time. In Heath, Jarrow and Morton (1990) it is demonstrated that the estimated factor-loadings can be used as a proxy for the volatility structure in a non-parametric Heath, Jarrow and Morton framework.

There is however a fundamental hitch in these analyses, since they do not take into account the dynamic part of the process – ie the resulting factor-pattern is neither directly nor indirectly analysed. This implies among other things that it is difficult to relate these factors to economic factors such as the yield spread, volatility or, for that matter, macroeconomic factors – which from the point of view of practive portfolio management must be considered rather important.

An alternative approach in order to solve this problem could be to take as one's starting point one of the multi-factor yield-curve models available, such as Longstaff and Schwartz (1990), Beaglehole and Tenney (1991) and Langetieg (1980) and thereafter estimate the model parameters under the assumption of time-homogeneity.

In this paper, however, our starting point is the observation that the estimated factor-loadings are to be considered a standardised volatility structure. More precisely, the objective of this paper is to analyse the implications for the existence of state variables in the arbitrage-based Heath, Jarrow and Morton framework.

The starting point here is the Gaussian framework due to its fine theoretical qualities which

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among other things means that it is possible to find analytical expressions for a number of interest rate derivative securities, see Musiela and Rutkowski (1997).

This paper will establish the relationship between the statistical and dynamic aspects of the linear factor structure taking a probability-based framework as the starting point. One of the main results of this paper is that the identification of the state variables gives rise to a Markovian stochastic system. This Markovian structure is set up by first assuming that the dynamics in the state variables follows a generalised Vasicek model (a multi-dimensional Hull and White) and second that the dynamics in the forward rates is defined in an m-factor model in the arbitrage-based framework.

This implies that the relation between the stochastic elements imposes restrictions on the mean-reversion matrix for the state variables, as was shown for the first time in literature by Karoui and Lacoste (1992).

A new method is also introduced for estimating volatility structures in the Heath, Jarrow and Morton framework, using the state-space terminology and the Kalman filter.

The paper consists of the following sections: In sections 2 and 3, we show the basic conditions pertaining to the dynamics of interest rates.

Then, in section 4, we set up the model framework. Here we show the relationship between the identification of state variables and the modelling of the dynamics in a forward rate structure. Thereafter, in section 5, we specifiy the linear factor structure and demonstrate that the time-dependent parameter in the Hull and White model is fully defined by the initial forward rates and the parameters that determine the volatility structure.

In section 6, we specify the restrictions on the mean-reversion matrix when state variables are introduced in the Heath, Jarrow and Morton framework.

Section 7 shows how the dynamic factor structure can be estimated by using the state-space model and the Kalman filter. In this section we will also specify a particular Markovian dynamic factor structure which we will assume represent a reasonable description of the dynamics in the yield-curve.

Then, in section 8, we will estimate the model we developed in section 7 for the period beginning 1990 to mid 1998, and thereafter consider the results.

2. General properties of the term structure of interest rates

In this paper I consider a continuous trading economy with zero-coupon bonds and a money market account with a trading interval $[0,\tau]$, for a fixed $\tau > 0$. In addition, it is assumed that money does not exist, i.e. that the agents in the economy are forced at all times to invest all their funds in assets. As usual, the uncertainty in the economy is characterized by the probability space (Ω,F,P), where Ω is the entire state space, P is a probability measure and F is the event space. At the same time, it is assumed that an m-dimensional Wiener process exists: $W = [W(t);0 < t \leq T < \tau]$, where the components $W_i(t)$, for $i = \{1,2,....,m\}$ are independent one-dimensional Wiener processes with a drift equal to zero (0) and a variance equal to one (1).

In addition, a continuum of zero-coupon bonds trade with different maturities T, for T ε [0, τ), where P(t,T) denotes the price at time t, for t ε [0,T), of a zero-coupon bond expiring at time T.

In addition, it is a condition that P(T,T) = 1, which means that at the maturity date the bond must have a value equal to the face value.

The instanteneous forward rate at time t, for T > t, $r^F(t,T)$ is defined by:

$$r^F(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \qquad \textit{for all } T \ \varepsilon \ [0,\tau] \textit{ and all } t \ \varepsilon \ [0,T] \tag{1}$$

From this it can be seen that $r^F(t,T)$ represents the rate that one can contract for at time t, for a risk-free investment in a forward contract that runs from time T to T + α, for α ≈ 0.

The relation given by formula 1 means that the price of a zero-coupon bond P(t,T) can be written as:

$$P(t,T) = \exp\left(-\int_t^T r^F(t,s)ds \right) \qquad \textit{for all } T \ \varepsilon \ [0,\tau] \textit{ and all } t \ \varepsilon \ [0,T] \tag{2}$$

In addition, the spot rate at time t is given by the instantaneous forward rate of a forward contract that runs from t to t + α, i.e.:

$$r(t) = r^F(t,t) \qquad \textit{for all } t \ \varepsilon \ [0,\tau] \tag{3}$$

The bond price process also entails the presence of a yield curve at any time t, which can be written as:

$$\begin{gathered} R(t,T) = -\frac{\ln P(t,T)}{T - t} \qquad \textit{for } t < T \leq \tau \\ \textit{for} \\ P(t,T) = \exp[-R(t,T)(T - t)] \qquad \textit{for } t < T \leq \tau \end{gathered} \tag{4}$$

From this formula it can be seen that R(t,T) is the continuous yield to maturity of a zero-coupon bond at time t over the period [t,T].

The forward rate process at time T^F, observed at time t, for $t < T^F < T \leq \tau$, for a zero-coupon bond with maturity at time T means that:

$$P(t,T^F,T) = \frac{P(t,T)}{P(t,T^F)} \qquad for\ t < T^F < T \leq \tau \tag{5}$$

In addition, this implies that the term structure of forward rates $R(t,T^F,T)$ at time t across the interval $[T^F,T]$, is defined by:

$$R(t,T^F,T) = -\frac{\ln P(t,T^F,T)}{T - T^F} \qquad for\ t < T^F < T \leq \tau \tag{6}$$

Where this means that the term structure of forward rates $r^F(t,T)$ is related to $R(t,T^F,T)$ in the following way:

$$r^F(t,T) = \lim_{(T^F \to T)} R(t,T^F,T) \qquad for\ t < T^F < T \leq \tau \tag{7}$$

Finally, the price process for the money market account (i.e., the value of a unit that has a growth/capitalization factor that is given by the risk-free rate) is given by the following relation:

$$M(t) = e^{\int_0^t r(s)ds} \qquad for\ all\ t\ \varepsilon\ [0,\tau] \tag{8}$$

In this economy, the tradeable assets are given by the zero-coupon bonds, the money market account and the various derived instruments that can be constructed.

3: Definition of the yield-curve dynamic

The dynamic in the zero-coupon bond-prices P(t,T), for $t < T \leq \tau$, is assumed to be governed by an Ito process under the risk-neutral martingale measure Q:

$$\frac{dP(t,T)}{P(t,T)} = rdt - \sum_{i=1}^{m} \sigma_P(t,T;i)d\tilde{W}_i(t) \quad for \quad \tilde{W}_i(t) = W_i(t) - \Gamma_i(t) \tag{9}$$

Where we have that P(0,T) is known for all T and P(T,T) = 1 for all T. Furthermore r is the risk-free interest rate, and $\sigma_P(t,T;i)$ represents the bond-price volatility, which can be associated with the i'th Wiener process, where $\tilde{W}_i$ is a Wiener process on (Ω,F,Q), for dQ = ρdP and ρ is the Radon-Nikodym derivative. We also have that $\Gamma_i(t)$ represents the market-price of risk that can be associated with the i'th Wiener process.

In order to derive the following results it is not necessary to assume that $\sigma_p(t,T;i)$ for i =

$\{1,2,...m\}$ is deterministic. It is sufficient to assume that $\sigma_p(t,T;i)$ is bounded, and its derivatives (which is assumed to exist) are bounded.

Formula 9 can be rewritten as:

$$d\ln P(t,T) = \left[r - \frac{1}{2}\sum_{i=1}^{m}\sigma_P^2(t,T;i)\right]dt - \sum_{i=1}^{m}\sigma_P(t,T;i)d\tilde{W}_i(t) \tag{10}$$

The solution to this process can be expressed as:

$$\ln P(t,T) = \ln P(0,T) + \int_0^t\left[r(s) - \frac{1}{2}\sum_{i=1}^{m}\sigma_P^2(s,T;i)\right]ds - \sum_{i=1}^{m}\int_0^t\sigma_P(s,T;i)d\tilde{W}_i(s) \tag{11}$$

and:

$$0 = \ln P(0,t) + \int_0^t\left[r(s) - \frac{1}{2}\sum_{i=1}^{m}\sigma_P^2(s,t;i)\right]ds - \sum_{i=1}^{m}\int_0^t\sigma_P(s,t;i)d\tilde{W}_i(s) \tag{12}$$

Where equation 12 follows from the horizon condition that P(T,T) = 1.

The drift in the process for the bond-price - r in formula 9 - can now be eliminated if we consider the difference between the process defined in formula 11 and the process that follows from the horizon condition (formula 12), ie:

$$\begin{aligned}\ln P(t,T) = \ln\frac{P(0,T)}{P(0,t)} &- \sum_{i=1}^{m}\int_0^t\frac{1}{2}[\sigma_P^2(s,T;i) - \sigma_P^2(s,t;i)]ds \\ &- \sum_{i=1}^{m}\int_0^t[\sigma_P(s,T;i) - \sigma_P(s,t;i)]d\tilde{W}_i(s)\end{aligned} \tag{13}$$

An expression for the yield-curve R(t,T) can now be derived by using equation 4:

$$\begin{aligned}R(t,T) = R^F(0,t,T) &+ \sum_{i=1}^{m}\int_0^t\frac{1}{2}\left[\frac{\sigma_P^2(s,T;i) - \sigma_P^2(s,t;i)}{T - t}\right]ds \\ &+ \sum_{i=1}^{m}\int_0^t\left[\frac{\sigma_P(s,T;i) - \sigma_P(s,t;i)}{T - t}\right]d\tilde{W}_i(s)\end{aligned} \tag{14}$$

The process for the forward-rates, can also be derived - namely by using formula 2 and 13, ie:

$$r^F(t,T) = r^F(0,T) + \sum_{i=1}^{m} \int_0^t \sigma^F(s,T;i)\sigma_P(s,T;i)ds + \sum_{i=1}^{m} \int_0^t \sigma^F(s,T;i)d\tilde{W}_i(s) \qquad \textbf{(15)}$$

Where $\sigma^F(t,T;i)$ is defined as $\dfrac{\partial \sigma_P(t,T;i)}{\partial T}$, and can be recognized as being a measure for the forward rate volatility.

We furthermore assume that the volatility function satisfies the usual identification hypothesis, that $\begin{pmatrix} \sigma^F(t,T_1) \\ \sigma^F(t,T_2) \\ . \\ . \\ \sigma^F(t,T_m) \end{pmatrix}$ is non singular for any t and any unique set of maturities $[T_1, T_2, \ldots T_m]$.

The spot-rate process is easily found from here:

$$r(t) = r^F(0,t) + \sum_{i=1}^{m} \int_0^t \sigma^F(s,t;i)\sigma_P(s,t;i)ds + \sum_{i=1}^{m} \int_0^t \sigma^F(s,t;i)d\tilde{W}_i(s) \qquad \textbf{(16)}$$

That is, the spot-rate process is identical to the forward-rate process, except that in formula 16 we have a simultaneous variation in the time-and maturity arguments.

It may be seen from formulas 15 and 16 that the process for the interest rates is fully defined by the initial yield-curve and the volatility structure, which is precisely the main result of the Heath, Jarrow and Morton (1991) model framework.

An alternative expression of the forward rate process may be found by introducing $\gamma = T - t$, where $\gamma \geq 0$, and γ is a constant. It can then be demonstrated that the forward rate $r^F(t, t + \gamma)$ satisfies the following stochatic differential equation:

$$\begin{aligned} dr^F(t,t+\gamma) = &\left[\frac{\partial r^F(t,t+\gamma)}{\partial \gamma} + \sum_{i=1}^{m} \sigma^F(t,t+\gamma;i) \int_t^{t+\gamma} \sigma^F(t,s;i)ds\right]dt \\ &+ \sum_{i=1}^{m} \sigma^F(t,t+\gamma;i)d\tilde{W}_i(t) \end{aligned} \qquad \textbf{(17)}$$

ie the risk-free drift is equal to the first derivative of the forward rate curve plus a volatility term.

Another way to express the process for the forward rates is to use the short rate as the horizon condition, so that:

$$r^F(t,T) = r(T) - \sum_{i=1}^{m} \int_t^T \sigma^F(s,T;i)\sigma_P(s,T;i)ds - \sum_{i=1}^{m} \int_t^T \sigma^F(s,T;i)d\tilde{W}_i(s) \qquad \textbf{(18)}$$

The only difference between formula 15 and formula 18 is that in formula 15, the yield-curve is an initial condition while in formula 18, the spot rate functions as a horizon condition.

Formula 17, however, stems from Brace and Musiela (1997). This expression represents the stochastic differential equation which drives forward rates with a fixed maturity.

If we now let T = t + γ and then differentiate $r^F(t,T)$ with respect to t, we get:

$$dr^F(t,t+\gamma) = \frac{\partial r^F(t,t+\gamma)}{\partial \gamma}dt + dr^F(t,T)_{|T=t+\gamma} \qquad \textbf{(19)}$$

From formula 15 we furthermore have:

$$dr^F(t,T) = \sum_{i=1}^{m} \sigma^F(t,T;i)\sigma_P(t,T;i)dt + \sum_{i=1}^{m} \sigma^F(t,T;i)d\tilde{W}_i(t) \qquad \textbf{(20)}$$

where formula 17 now follows directly.

We may therefore conclude that this model framework has been constructed in such a way that the observed initial the yield-curve is correctly specified, since it is used as input in the model specification itself.

This contrasts with the traditional method by which the dynamics of the yield-curve is defined, such as in Cox, Ingersoll and Ross (1985), Vasicek (1977), Longstaff and Schwartz (1990) and Beaglehole and Tenney (1991). It is namely so that the process for one or several pre-defined state variables is specified here, in which the state variables fully define both the initial the yield-curve and the inner dynamics. When the parameter vector that describes the dynamics in the state variables is known, the initial yield-curve and its dynamics are fully specified here, but this is no general guarantee that the actual initial interest rate structure is identical with that specified by the model. In practice, there are two different methods for bringing these models into agreement with the initial yield-curve. The unknown parameter vector can be identified either by using the implicit volatility approach from Brown and Schaefer (1994) or by introducing time-dependent parameters in the process for the state variables, as in Hull and White (1990).

In the following, we will assume that the volatility structure $\sigma^F(t,T)$ is a deterministic function of time t and maturity T. As a consequence, the dynamics of the yield-curve is Gaussian.

4: Establishment of Model Framework

It is well known that a model for the dynamics in the yield-curve can be contained within a process for the spot rate This is of course obvious in the case of models such as Vasicek and Cox, Ingersoll and Ross.The classical method is also to represent the price of a zero-coupon bond as the risk-neutral expected value of the spot rate process, so that:

$$P(t,T) = \tilde{E}_t\left[\exp\left(-\int_t^T r(s)ds\right)\right] \qquad \textbf{(21)}$$

This clearly shows that the problem concerning the determination of bond prices can be reduced to a modelling of the spot-rate process.

In the case where a one-factor model is under consideration, it is obvious that the spot rate will constitute a state-variable in its own right. If however, an m-factor model is analysed, it is also obvious that the spot rate cannot be Markovian. On the other hand, the number of necessary variables corresponds to the minimal dimension of a Markov process, which can be identified with a vector of state variables.

As the ability to identify the chosen state-variables with observable economic factors is of fundamental importance, we will only focus on “observable state variables”.

Assumption no. 1

The dynamics in the yield-curve can be described by an m-dimensional factor structure, if there exists an m-dimensional process Z_t, and a deterministic function $F(t,T,Z_t)$, so that:

$$r^F(t,T) = F(t,T,Z_t) \quad \textit{for all } t \leq T < \tau \qquad \textbf{(22)}$$

Note that this assumption could just as well have been formulated with spot yields as the starting point (or bond prices for that matter), since there is a clear connection between forward rates and spot yields. Spot yields can be derived, as is well known, from forward rates via a simple integration.

Taking into consideration the relationship in formula 22, it is possible for the factor structure to be defined by both endogenous and exogenous factors. Interest rates and interest rate spreads are typical engodenous factors while interest rate volatility or economic factors are examples of exogenous factors[2].

In general, however, there is always an endogenous representation which means that the factors can be identified with some particular interest rates.

Assumption no. 2

The stochastic process for this m-dimensional vector of observable state variables Z_t may be found as the solution to the following multi-dimensional Ornstein-Uhlenbeck process:

$$dZ_t = (\Phi(t) + BZ_t)dt + S[dW - \lambda dt] \tag{23}$$

where S is a mxm matrix of diffusion coefficients, dW is a vector of m elements and B is a mxm matrix, where it is further assumed that B and S are time-homogeneous. Furthermore, Φ is a deterministic function of t and λ is a vector of market price of risk parameters.

For this process we have that the volatility structure for the forward rates $\sigma^F(t,T)$ and bond prices $\sigma_P(t,T)$ can be expressed as follows:

$$\begin{aligned} \sigma^F(t,T) &= \omega^T e^{-B(T-t)} S \\ \sigma_P(t,T) &= \omega^T \int_t^T \sigma^F(t,s)ds = \omega^T B^{-1}[I - e^{-B(T-t)}]S \end{aligned} \tag{24}$$

In particular we have that $r(t) = \omega^T Z_t$, where ω is an m-dimensional Boolian column vector with at least oné 1 and otherwise zeros (0).

The model expressed by formula 23 is identical with the specification in Karoui and Lacoste (1992). Furthermore, the model also fits in with the more general model specification from Duffie and Kan (1996).

A rather more general expression of the S matrix shows, however, the precise difference between the Duffie and Kan model and the model expressed under Assumption no. 2.

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[2] Although it is obvious that the spot rate can be a state variable, it does not mean that it has to be a state variable; see, for example, the Walter (1995) model where the spot rate is not one of the state variables. However, the spot rate is given by a linear combination of state variables.

Let us in that connection express the matrix S as:

$$S = \begin{bmatrix} \sqrt{a_1 + b_1 X_1(t)} & 0 & . & 0 \\ S_{21} & R_{22}\sqrt{a_2 + b_2 X_2(t)} & . & 0 \\ . & . & . & . \\ S_{m1} & S_{m2} & . & R_{mm}\sqrt{a_m + b_m X_m(t)} \end{bmatrix}. \qquad \textbf{(25)}$$

where S is a lower triangular matrix, which can be found by applying Choleski factorisation on the variance matrix SS^T. Furthermore, Rii, for i = {2,3,...,m}represents an adjustment parameter that arises in connection with factorisation and is related to the coefficient of correlation.

Assuming that the vector bi = 0, for i = {1,2,....,m} we get the model specified under Assumption no. 2. The Duffie and Kan general model appears for bi ≠ 0, where vector b generates a form for "stochastic volatility"[3].

Proposition no. 1

The forward rate is a linear function of Z_t for every t and T ≥ t:

$$r^F(t,T) = \omega^T e^{-B(T-t)} Z_t + \omega^T \int_t^T e^{-B(T-s)} \Phi(s) ds - \int_t^T \sigma_P(s,T)^T \sigma^F(s,T) ds \qquad \textbf{(26)}$$

and for the initial forward curve (t = 0) we find that:

$$r^F(0,T) = \omega^T e^{-BT} Z_0 + \omega^T \int_0^T e^{-B(T-s)} \Phi(s) ds - \int_0^T \sigma_P(s,T)^T \sigma^F(s,T) ds \qquad \textbf{(27)}$$

Proof:

From formula 18 we have:

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[3] The Duffie and Kan model can be recognized as a extended multi-dimensional CIR-model, where in order to get an analytical solution it is necassary to assume that there is no correlation between the Brownian motions - ie. the S matrix is a diagonal matrix. For further infomation see Duffie and Kan (1996) and Madsen (1998a) where multi-factor state-variable models has been analysed.

$$r^F(t,T) = r(T) - \int_t^T \sigma_P(s,T)^T \sigma^F(s,T)ds - \int_t^T \sigma^F(s,T)^T d\tilde{W} \tag{28}$$

Since $r(T) = \omega^T Z_T$ this can be expressed as:

$$r^F(t,T) = \omega^T Z_T - \int_t^T \sigma_P(s,T)^T \sigma^F(s,T)ds - \int_t^T \sigma^F(s,T)^T d\tilde{W} \tag{29}$$

Furthermore we know that Z_T for $T \geq t$, can be written as:

$$Z_T = e^{-B(T-t)} Z_t + \int_t^T e^{-B(T-s)} \Phi(s)ds + \int_t^T e^{-B(T-s)} S d\tilde{W}(s) \tag{30}$$

If formula 30 is inserted in formula 29, we find:

$$\begin{aligned} r^F(t,T) &= \omega^T e^{-B(T-t)} Z_t + \omega^T \int_t^T e^{-B(T-s)} \Phi(s)ds + \omega^T \int_t^T e^{-B(T-s)} S d\tilde{W}(s) \\ &\quad - \int_t^T \sigma_P(s,T)^T \sigma^F(s,T)ds - \int_t^T \sigma^F(s,T)^T d\tilde{W}(s) \\ &= \omega^T e^{-B(T-t)} Z_t + \omega^T \int_t^T e^{-B(T-s)} \Phi(s)ds - \int_t^T \sigma_P(s,T)^T \sigma^F(s,T)ds \end{aligned} \tag{31}$$

Qed.

Where the last term in formula 31 (and with it formula 26) can be recognized as being an expression for the variance in $\int_t^T r(s)ds = \omega^T \int_t^T Z_s ds$ as $\int_t^T \sigma_P(s,T)^T \sigma^F(s,T)ds = \frac{1}{2}\int_t^T \partial_T \|\sigma_P(s,T)\|^2 ds$.

5: Specification of the linear factor structure

From now on we will consider forward rates with a fixed time to maturity instead of a fixed maturity - which is normally the case. We will therefore use the Brace and Musiela (1997) approach when analysing the dynamic in the forward rates, see section 3.

More precisely $\sigma^F(t,T)^T$ is under this assumption defined as follows:

$$[\sigma^F(t,t+\gamma]),\ \sigma^F(t,t+2\gamma),........,\ \sigma^F(t,t+m\gamma)]^T \qquad \textbf{(32)}$$

from which we have that the forward rates are equally distributed over the maturity spectrum. From equation 32 it is obvious that the volatility structure is separable in time and maturity. As it is given that Z_t is a Gaussian process, this means that the forward rates are also Gaussian. As a consequence, the relationship between the forward rates and Z_t is affine, thus:

$$r^F(t,T) = b(T-t)^T Z_t + a(t,T) \quad for\ alle\ t \le T < \tau \qquad \textbf{(33)}$$

where b(T-t) and a(t,T) are deterministic functions of t and T - where it follows that formula 33 is a consequence of the Markov property.

It is furthermore defined that $\Lambda(T-t) = \int_0^{T-t} b(s)ds$, and where we have that $\Lambda(0) = 0$ and $b(0) = 1$.

If we wish to identify the state variables, we are obliged to let a(t,T) be a 0-vektor and $b(T\text{-}t)^T$ be a Boolean diagonal matrix.

If, alternatively, we wish to express the linear factor structure in the yield-curve, we have:

$$R(t,T) = \frac{1}{T-t}\Lambda(T-t)^T Z_t + \frac{1}{T-t}\int_t^T a(t,s)ds \quad for\ all\ t \le T < \tau \qquad \textbf{(34)}$$

The main objective of this section is to determine the characteristics of the the yield-curve and the dynamics of the underlying state variables implied by this factor structure.

The result of this and the next section will demonstrate that the assumption that there are one or more state variables that describe the dynamics of the yield-curve is sufficiently restrictive to determine the class of permissable volatility functions.

It may be deduced that the total equation system for describing the linear factor structure can be expressed by the following three (3) relationships:

$$\begin{cases} r^F(t,T) = b(T-t)^T Z_t + a(t,T) \\ r^F(t,T) = r^F(0,T) + \int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds + \int_0^t \sigma^F(s,T)^T d\tilde{W}(s) \\ Z_t = Z_0 + \int_0^t D(s)ds + \int_0^t S d\tilde{W}(s) \end{cases} \quad \textbf{(35)}$$

where $r^F(t,T)$ and a(t,T) are vectors of m-elements, b(T-t) is an m x m-matrix, Z_t and D(t) are vectors of m-elements, and S is a lower triangular matrix of the m x m dimension. Furthermore, $\sigma^F(t,T)$ and $\sigma_P(t,T)$ are vectors of the m dimension. We also find that D(t) is a vector of m-elements[4].

These relationships are now made consistent by relating the parameters in the individual stochastic integrals to each other, as follows:

$$\begin{cases} \sigma_P(t,T)^T \sigma^F(t,T) = b(T-t)^T D(t) + \dfrac{\partial a(t,T)}{\partial t} - \left(\dfrac{\partial b(T-t)}{\partial t}\right)^T Z_t \\ \sigma^F(t,T) = S^T b(T-t) \\ \sigma_P(t,T) = S^T \Lambda(T-t) \end{cases} \quad \textbf{(36)}$$

These 3 equations in formula 36 indicate that:

$$\Lambda(T-t)^T S S^T b(T-t) = b(T-t)^T D(t) + \frac{\partial a(t,T)}{\partial t} - \left(\frac{\partial b(T-t)}{\partial t}\right)^T Z_t \quad \textbf{(37)}$$

Formula 37 applies for all T $\geq$ t and results in an m-dimensional vector, which indicates that SS^T and D(t) are affine functions of Z_t,, provided that this equation can be inverted.

If this is the case, it means that S can be written in the following form:

[4] Even though we assume here that the number of factors is equal to the dimension of the Wiener process, this can be relaxed - as will appear later in this paper.

$$S = \begin{bmatrix} S_{11} & 0 & . & 0 \\ S_{21} & S_{22} & . & 0 \\ . & . & . & . \\ S_{m1} & S_{m2} & . & S_{mm} \end{bmatrix}. \tag{38}$$

where S is found, as was mentioned above, in a Choleski factorisation.

Furthermore, it is evident from formula 23 that D(t) can be written in the following manner:

$$D(t) = \Phi(t) + BZ_t \tag{39}$$

Furthermore, it can be deduced that $\Lambda(T-t)$ and b(T-t) respectively can be expressed as:

$$\begin{aligned} \Lambda(T-t) &= (S^{-1})^T \sigma_P(t,T) \\ b(T-t) &= (S^{-1})^T \sigma^F(t,T) \end{aligned} \tag{40}$$

Furthermore, by inserting formula 40 into formula 36, we get:

$$\sigma_P(t,T)^T \sigma^F(t,T) = b(T-t)^T [\Phi(t) + BZ_t] + \frac{\partial a(t,T)}{\partial t} - \left(\frac{\partial b(T-t)}{\partial t} \right)^T Z_t \tag{41}$$

Which mean that the function Λ(t,T) is the solution to the following equation system[5]:

$$b(T-t) = b(t-t) + B^T \Lambda(T-t) \tag{42}$$

An expression for a(t,T), ie the point of intersection, can also be found by using formula 41. By integrating the ordinary differential equation for a(t,T) while taking into consideration a(T,T) = 0, we arrive at the following result:

$$a(t,T) = \int_t^T b(T-s)^T \Phi(s) ds - \int_t^T \sigma_P(s,T)^T \sigma^F(s,T) ds \tag{43}$$

Formula 42 can be recognized as an ordinary first order differential equation with a solution of the following form:

---

[5] In the more general setup from Duffie and Kan $\Lambda(T-t)$ will be the solution to a multi dimensional Ricatti equation. However, in our case this degenerates into the relationship in formula 42, because in formula 25 we have that $b_i = 0$, for i = {1,2,...,m}.

$$\Lambda(T - t) = B^{-1}[I - e^{-B(T - t)}] = QA^{-1}[I - e^{-A(T - t)}]Q^{-1} \tag{44}$$

where Q is a matrix with B´s eigenvectors in the columns and A is a diagonal matrix with B´s eigenvalues in the diagonal[6]. This therefore means that the matrix of factor loadings $b(T-t)^T$ can be written in the following form:

$$b(T - t)^T = e^{-B(T - t)} = Qe^{-A(T - t)}Q^{-1} \tag{45}$$

where this very same relationship is used in the paraphrase from formulas 41 to 42.

All in all, this means that the linear factor structure can be written in the following form:

$$\begin{gathered} r^F(t,T) = b(T - t)^T Z_t + a(t,T) \\ \textit{for} \\ b(T - t)^T = e^{-B(T - t)} \\ \textit{and} \\ a(t,T) = \int_t^T e^{-B(T - s)}\Phi(s)ds - \int_t^T \sigma_P(s,T)^T \sigma^F(s,T)ds \end{gathered} \tag{46}$$

where this expression (if we disregard the vector ω) is seen to be identical with formula 26 from Proposition no. 1, where it is shown that the forward rate is a linear function of Z_t.

In connection with the above, it is of interest to specify the functional form of the time-dependent drift parameter Φ(t).

Proposition no. 2

Φ(t) can be expressed as follows:

$$\begin{gathered} D(t) = \Phi(t) + BZ_t \\ \textit{for} \\ \Phi(t) = \partial_t \alpha(t) - B\alpha(t) \\ \textit{and} \\ \alpha(t) = r^F(0,T) + \int_0^t \sigma^F(s,T)^T \sigma_P(s,T)ds \end{gathered} \tag{47}$$

Proof:

[6] Of course this relationship is only true under the assumption that the matrix B is non-singular. In the case of a non-singular B matrix then other methods - than eigenvalue decomposition - have to be used when integrating the matrix exponential function, see Golub and Van Loan (1993).

From formula 34 we have:

$$
\begin{aligned}
& r^F(0,T) + \int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds + \int_0^t \sigma^F(s,T)dW = b(T-t)^T Z_0 \\
& + \int_0^t b(T-s)^T \Phi(s)ds + \int_0^t b(T-s)^T BZ_s ds + \int_0^t S^T b(T-s)dW + a(t,T)
\end{aligned}
\tag{48}
$$

which can be rewritten as:

$$
\begin{aligned}
& r_t^F(0,T) + \partial_t\left[\int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds\right] + \int_0^t \left[\partial_t \sigma^F(s,T) - S^T \partial_t b(T-s)\right]dW \\
& = \partial_t b(T-t)^T Z_0 + b(T-t)^T \Phi(t) + b(T-t)^T BZ_t + \partial_t a(t,T)
\end{aligned}
\tag{49}
$$

If we now wish to identify the state variables that describe the dynamics of the yield-curve, and assume in this connection that the dynamics are described by an m-factor model, formula 49 degenerates into:

$$
\Phi(t) = r_t^F(0,T) + \partial_t\left[\int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds\right] + \int_0^t \partial_t \sigma^F(s,T)dW - BZ_t
\tag{50}
$$

As it is known that $\int_0^t \sigma^F(s,T)dW = Z_t - r^F(0,T) + \int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds$ *formula 50 can be rewritten as:*

$$
\begin{aligned}
& \Phi(t) = r_t^F(0,T) + \partial_t\left[\int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds\right] - BZ_t - B\left[r^F(0,T) + \int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds - Z_t\right] \\
& \Rightarrow \Phi(t) = r_t^F(0,T) + \partial_t\left[\int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds\right] - B\left[r^F(0,T) + \int_0^t \sigma_P(s,T)^T \sigma^F(s,T)ds\right]
\end{aligned}
\tag{51}
$$

which completes the argument since formula 47 follows as a direct consequence from this.

Qed.

It should however be pointed out that if we only observe a one-factor model, the expression in formula 47 degenerates into:

$$
\begin{gathered}
D(t) = \Phi(t) + \kappa r(t) \\
\textit{for} \\
\Phi(t) = \partial_t \alpha(t) - \kappa\alpha(t) \\
\textit{og} \\
\alpha(t) = r^F(0,t) + \frac{\sigma^2}{2\kappa^2}(1 - e^{-\kappa t})^2
\end{gathered}
\qquad \textbf{(52)}
$$

because $\int_0^t \sigma_P(s,t)\sigma^F(s,t)ds = \frac{1}{2}\int_0^t \partial_t \sigma_P^2(s,t)ds = \frac{1}{2}\sigma_P^2(0,t).$

Where D(t) in formula 52 is recognized as being identical to the drift specification in the Hull and White model and where α(t) in the Hull and White model functions as the parameter that displaces the nodes in the trinomial lattice - ie α(t) is the expected short-futures rate[7]. Thus, in a one-factor version, the model degenerates into a Hull and White model, ie an Extended Vasicek model.

From this we can draw the conclusion that the time-dependent drift parameter is fully specified by the initial yield-curve and the volatility structure.

6: Identification of state variables

The definition of the vector of the state variables Z_t is the only condition we have not yet dealt with. In this section, we wish to show that the definition of the state variables has an effect on the factor structure from formula 46 via restrictions on the mean reversion matrix B.

Proposition no. 3[8]

The m-dimensional process $Z_t(x)$ can be identified by the successive derivatives of the forward rate curve for a given maturity x ε {γ,2γ,..., mγ}:

[7] See Hull and White (1994).

[8] The outline of this proof relies on Karoui and Lacoste (1992).

$$Z_t(x) = \begin{pmatrix} r^F(t,t+x) \\ \frac{\partial r^F(t,t+x)}{\partial x} \\ . \\ . \\ . \\ \frac{\partial^{m-1} r^F(t,t+x)}{\partial x^{m-1}} \end{pmatrix} \tag{53}$$

the stochastic process for $Z_t(x)$ is driven by the following generalized Vasicek process:

$$dZ_t(x) = [\Phi(t) + BZ_t(x)]dt + S[dW - \lambda dt] \tag{54}$$

where B is defined as:

$$B = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & . & . & 0 & 1 \\ \alpha_1 & \alpha_2 & . & . & \alpha_m \end{pmatrix} \tag{55}$$

and where $\frac{\partial^i r^F(t,t+x)}{\partial x^i}$ for i = {0,1,....,m-1} represents the i´th derivative of the forward rate curve with respect to the maturity date x.

Proof:

Formula 17 gives us the stochastic differential equation that is defined by the first component in $Z_t(x)$. More precisely we have that the drift of $r^F(t,t+x)$ is given by the sum of a volatility term (which is a deterministic function of time) and by $\frac{\partial r^F(t,t+x)}{\partial x}$, which is equal to the second component in $Z_t(x)$. From here we can deduce that the first row in the matrix B is equal to [0,1,.....,0].

As regards the next m-2 rows, their validity is indicated by the fact that formula 17 also applies to the next m-2 components of $Z_t(x)$.

When formula 17 is differentiated with regard to x, for i = 1,2,....,m-1 we find the following expression:

$$d\frac{\partial^i r^F(t,t+x)}{\partial x^i} = \left[\frac{\partial^{i+1} r^F(t,t+x)}{\partial x^{i+1}} + \partial_x^i \sigma_P(t,t+x)^T \sigma^F(t,t+x)\right]dt + \partial_x^i \sigma^F(t,t+x)^T d\tilde{W}(t) \quad \textbf{(56)}$$

from which the first m-1 rows of B follows.

As is seen from formula 56, the drift in $\partial_x^i r^F(t,t+x)$ *is dependent of* $\partial_x^{i+1} r^F(t,t+x)$*. In this connection, formula 46 suggests the existence of a linear relationship between the forward rates and the vector of the state variables (where this vector of state variables is assumed to be given by* $Z_t(x)$*) ie :*

$$r^F(t,t+x) = b(x)^T Z_t(x) + a(t,t+x) \quad \textbf{(57)}$$

If this relationship is differentiated m times, we find:

$$\frac{\partial^m r^F(t,t+x)}{\partial x^m} = \left(\frac{\partial^m b(x)}{\partial x^m}\right)^T Z_t(x) + \frac{\partial^m a(t,t+x)}{\partial x^m} \quad \textbf{(58)}$$

This expression implies that the functions α_i *for i = {0,2,....,m-1} are defined by* $-\frac{\partial^i b(x)}{\partial x^i}$*, where the last row of B now follows from here.*

We can thus conclude that the restrictions on the mean revision matrix result in a Frobenius matrix.

Qed.

Alternatively, instead of defining the vector of the state variables $Z_t(x)$ as the successive derivatives of the forward rate curve with respect to a particular maturity (in the above example x), we could have chosen to define Z_t, as $Z_t^T = [r^F(t,t+\gamma), r^F(t,t+2\gamma), \ldots, r^F(t,t+m\gamma)]$ and x ε {γ,2γ,..., mγ}. That is the state-variables are themselves forward rates.

However, the consequence is that the restrictions on the mean reversion matrix B are made indirectly via restrictions on e^{Bx}. To be more precise, Karoui and Lacoste have shown that there exists an β_i for i = {1,2,....,m} so that e^{Bx} is defined by:

$$e^{Bx} = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & . & . & 0 & 1 \\ \beta_1 & \beta_2 & . & . & \beta_m \end{pmatrix} \tag{59}$$

This shows that the restrictions on the B matrix are of a considerably more complex form.

Using our knowledge of the process for $Z_t(x)$ from formula 54, it is possible to determine the dynamics of any interest rate. For example, if we set x = 0, we find that the dynamics of the short rate may be expressed as $r(t) = [1,0,......,0]Z_t(x)$.

More generally, we find that there exists the following connection between the forward rates $r^F(t,t + x)$, for all x, and the process Z_t:

$$r^F(t,t + x) = \omega^T e^{-B(x-\gamma_0)} Z_t(x) + a(t,t + x) \tag{60}$$

where $\omega^T = [1,0,.....,0]$ and $\gamma_0 = 0$.

Taking into account formula 59 and the definition of the state variables we find the following expression for the forward rate with a maturity equal to x:

$$\begin{aligned} r^F(t,t + x) &= r^F(t,t + \gamma_o) + (x - \gamma_o)\frac{\partial}{\partial_x} r^F(t,t + \gamma_o) + ... \\ &+ \frac{1}{(m - 1)!}(x - \gamma_o)^{m-1}\frac{\partial^{m-1}}{\partial_x^{m-1}} r^F(t,t + \gamma_o) \end{aligned} \tag{61}$$

which implies a deterministic Taylor expansion over the entire yield-curve.

It should be pointed out here that the conditions that must be fulfilled before the model will satisfy the initial bond prices P(0,T) for T $\geq$ 0 are given in formula 27. We also find here that it is only when the volatility structure is known that the time-dependent drift parameter can be completely determined.

Actually we have from formula 56 that the m-1 first elements in $\Phi(t)$ can be expressed as:

$$[\sigma_P(t,t + x)^T\sigma^F(t,t + x),\ \partial_x\sigma_P(t,t + x)^T\sigma^F(t,t + x), ...,\ \partial_x^{m-2}\sigma_P(t,t + x)^T\sigma^F(t,t + x)] \tag{62}$$

This means that only the m´th component in Φ(t) remains to be determined

From formula 27 we can deduce the following relationship.

$$\omega^T \int_0^x e^{-B(x-s)} \left[\Phi_*(s) + \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ \Phi_m(s) \end{bmatrix} \right] ds = r^F(0,x) + \int_0^x \sigma_P(s,x)^T \sigma^F(s,x) ds - \omega^T e^{-Bx} Z_0(x) \tag{63}$$

Where $\Phi_*(t)^T = [\sigma_P(t,t+x)^T\sigma^F(t,t+x), \partial_x\sigma_P(t,t+x)^T\sigma^F(t,t+x), \ldots\ldots, \partial_x^{m-1}\sigma_P(t,t+x)^T\sigma^F(t,t+x)]$ and $\Phi_m(t)$ is defined is such a manner that formula 63 is valid for all x > 0. From this we therefore have that $\Phi(t) = \Phi_*(t) + \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ \Phi_m(t) \end{bmatrix}$.

From this result we can deduce that the initial bond price gives us no information about the volatility structure, because the model allows for time heterogenity. This issue will be addressed further in section 7, as it entails estimation problems.

Let us now recall the expression for the linear factor structure from formula 46, as follows:

$$r^F(t,t+x) = \omega^T e^{-Bx} Z_t(x) + a(t,t+x)$$
$$for$$
$$\begin{aligned} a(t,t+x) &= \int_t^{t+x} e^{-B(t+x-s)} \Phi(s) ds - \int_t^{t+x} \sigma_P(s,t+x)^T \sigma^F(s,t+x) ds \\ &= \int_0^x e^{-B(x-s)} \Phi(s) ds - \int_0^x \sigma_P(s,x)^T \sigma^F(s,x) ds \end{aligned} \tag{64}$$

where this is the so-called static part of the factor structure. The factor structures which have been analysed in literature are mainly those which are identical in appearance with the static part, see for example Litterman and Scheinkman (1988).

In Appendix A we show that the functional form of $b(T\text{-}t)^T$ (factor loadings matrix), considering the fact that all κ_i´s are unique, has the following appearance:

$$b(T - t)^T = \begin{pmatrix} e^{-\kappa_1\gamma} & e^{-\kappa_2\gamma} & . & . & e^{-\kappa_m\gamma} \\ e^{-\kappa_1 2\gamma} & e^{-\kappa_2 2\gamma} & . & . & e^{-\kappa_m 2\gamma} \\ . & . & . & . & . \\ e^{-\kappa_1 m\gamma} & e^{-\kappa_2 m\gamma} & . & . & e^{-\kappa_m m\gamma} \end{pmatrix} \quad \mathit{for}\ T - t\ \varepsilon\ [\gamma,2\gamma,.......,m\gamma] \tag{65}$$

where the process is stationary if all the κ_i´s are greater than zero (0).

Since this expression is independent of whether the vector of the state-variables is Z_t or $Z_t(x)$ we will in the following denote the vector of the state-variables as F_t, so that formula 64 takes on the following form:

$$r^F(t,t + k\gamma) = \begin{pmatrix} e^{-\kappa_1\gamma} & e^{-\kappa_2\gamma} & . & . & e^{-\kappa_m\gamma} \\ e^{-\kappa_1 2\gamma} & e^{-\kappa_2 2\gamma} & . & . & e^{-\kappa_m 2\gamma} \\ . & . & . & . & . \\ e^{-\kappa_1 m\gamma} & e^{-\kappa_2 m\gamma} & . & . & e^{-\kappa_m m\gamma} \end{pmatrix} F_t + a(t,t + k\gamma) \tag{66}$$

with respect to $a(t,t + k\gamma)$ we have the following expression:

$$a(t,t + k\gamma) = \int_0^{k\gamma} b(k\gamma - s)^T \Phi(s) ds - \int_0^{k\gamma} \sigma_P(s,k\gamma)^T \sigma^F(s,k\gamma) ds \tag{67}$$

We are now able to conclude that F_t (for the present factor structure) is independent of $\Phi(t)$ as it is included in the expression for $a(t,t + k\gamma)$, which means that the dynamic part of the factor structure can be written as:

$$
\begin{gathered}
dF_t = -DF_t dt + b(0,t)^T S d\tilde{W} \\
\textit{for} \\
D = \begin{pmatrix} e^{-\kappa_1 t} & 0 & . & . & 0 \\ 0 & e^{-\kappa_2 t} & 0 & . & 0 \\ . & . & . & . & . \\ 0 & . & . & 0 & e^{-\kappa_m t} \end{pmatrix} \\
\textit{and} \\
b(t)^T S = \begin{pmatrix} \sigma_1 e^{-\kappa_1 t} & 0 & . & . & 0 \\ 0 & \sigma_2 e^{-\kappa_2 t} & 0 & . & 0 \\ . & . & . & . & . \\ 0 & . & . & 0 & \sigma_m e^{-\kappa_m t} \end{pmatrix}
\end{gathered}
\tag{68}
$$

Alternatively, this expression can be written as:

$$
F_t = e^{-Dt} F_0 + \int_0^t b(t-s)^T S d\tilde{W} \tag{69}
$$

which means that the econometrics model (given unique ki´s) can be written as:

$$
\begin{gathered}
\text{Static Part} \quad r^F(t,t+k\gamma) = b(T-t)^T F_t + a(t,t+k\gamma) \\
\textit{for} \\
a(t,t+k\gamma) = \int_0^{k\gamma} b(k\gamma - s)^T \Phi(s) ds - \int_0^{k\gamma} \sigma_P(s,k\gamma)^T \sigma^F(s,k\gamma) ds \\
\text{Dynamic Part} \quad F_t = e^{-Dt} F_0 + \int_0^t b(t-s)^T S d\tilde{W}
\end{gathered}
\tag{70}
$$

There is however still a condition in formula 70 which makes this linear factor structure rather difficult to handle, ie the time-dependent parameter $\Phi(t)$.

However, by investigating the relationship from formula 63 we can determine the role played by the time-dependent parameter $\Phi(t)$, more precisely we can express $a(t,t + k\gamma)$ as:

$$
\begin{aligned}
a(t,t+k\gamma) &= -\int_0^{k\gamma} \sigma_P(s,k\gamma)^T \sigma^F(s,k\gamma)ds + r^F(0,k\gamma) + \int_0^{k\gamma} \sigma_P(s,k\gamma)^T \sigma^F(s,k\gamma)ds \\
&\quad - b(k\gamma)^T F_0 = r^F(0,k\gamma) - b(k\gamma)^T F_0
\end{aligned} \tag{71}
$$

Where it from here follows that a(t,t + kγ) (through the time-dependent parameter Φ(t)) captures the misspecification of the initial yield-curve. That is a(t, t + kγ) functions in principle as a correction term.

Another interesting observation is that a way to minimize the error term, is just to add additional state-variables - more precisely we have an exact description of the initial yield-curve if we introduce m state-variables.

Finally, we find that the total dynamic factor structure can be written in the following form:

$$
\begin{aligned}
\text{Static Part} \quad & r^F(t,t+k\gamma) = r^F(0,k\gamma) + b(T-t)^T[F_t - F_0] \\
\text{Dynamic Part} \quad & F_t = e^{-Dt}F_0 + \int_0^t b(t-s)^T S d\tilde{W}
\end{aligned} \tag{72}
$$

It is worth pointing out here that if the model is time homogenous[9] then we can deduce that the static part indicates that for a future date $T_1 > t$, we have that the function a(t,) does not depend on time anymore - that is the future forward rates are simple functions of the initial conditions.

I will for a moment dwell on this observation.

Let us first consider the case of time homogenity. In this case the forward rates are given by the state-variables F_t, ω. B and S[10]. From this we can deduce that if we fit the model to actual forward rates $r^F(t,t + k\gamma)$ for all k, then this will provide us with a simultaneous estimation of all the unknown parameters. From this it can be concluded that the observable yield-curve provide us with information about the volatility structure - more precisely it makes the volatility parameters S observable.

This is in contrast with the time heterogenity case - as we here have that the observable yield-curve only provides us with information through the time dependent funcion Φ(t), see Proposition no. 3. From this we can deduce that in the case of time heterogenity then the market data does not provide us with any information about the other parameters that govern the state-variables F_t, for example the volatility parameter S is in this case not observable.

[9] Time homogenity implies that a(t,) does not depend on time t - it is only a function of the time to maturity.

[10] This follows directly by the time homogenity assumption and formula 31.

This observation will influence how we from a practical estimation point of view decide to use formula 72 - this will be elaborated in section 7.1.

7: Estimation of the dynamic factor structure

The initial points in time are t = 1,2,....L , ie we assume that the yield-curve is observed at L dates with a fixed length between the observation points. The maturity points T are also made discrete with a pre-defined length $\gamma : T = t + \gamma, t + 2\gamma, ..., t + m\gamma$. Furthermore, we define a number k = 1,2,....m.

The factor structure is observed via the yield-curve defined as follows:

$$R(t,t + k\gamma) = \frac{1}{k\gamma} u_k^T F_t \tag{73}$$

where $u_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, ..., u_m = \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{pmatrix}$.

Let us now specify G as the m times m matrix, where each row is defined as $\frac{1}{\gamma} u_1^T, \frac{1}{2\gamma} u_2^T, ..., \frac{1}{m\gamma} u_m^T,$ so that the m-dimensional vector of variables Y_t is defined as: $Y_t = GF_t$[11].

In this connection, the dynamic factor structure takes on the following form:

$$\begin{aligned} &\text{Static Part} \quad R(t,t + k\gamma) = R(0,k\gamma) + Tb(k\gamma)^T [F_t - F_0] \\ &\text{Dynamic Part} \quad F_t = e^{-Dt} F_0 + \int_0^t b(t - s)^T S d\tilde{W} \end{aligned} \tag{74}$$

The factor structure from formula 74 is however defined under the risk-neutral probability measure Q.

[11] It is worth pointing out that this is a “summation” approximation of the integral that relates b(x) to Λ(x)/x.

From the point of view of estimation technique, however, it is necessary to work under the original probability measure P, since the process under the probability measure Q is irrelevant in this context. In order to make the model usable in practice, it is also necessary to assume that the market price of risk parameters is time-homogenous.

Under the original probability measure P, the factor structure takes on the following appearance:

$$\begin{aligned} &\text{Static Part} \quad R(t,t+k\gamma) = R(0,k\gamma) + Tb(k\gamma)^T[F_t - F_0] \\ &\text{Dynamic Part} \quad F_t = e^{-Dt}F_0 - \int_0^t b(t-s)^T \lambda S^T ds + \int_0^t b(t-s)^T S dW \end{aligned} \tag{75}$$

In connection with formula 75, there are a couple of conditions which must be taken into account. First we have to determine an estimation technique, and second we have to define the dimension of the vector of state-variable F_t.

Let us deal with these problems in the named order:

7.1 State Space formulation of the dynamic factor structure [12]

In this section, we will demonstrate how state space formulation combined with Kalman filter algorithm can be used to calculate the maximum likelihood function for the observed rates of interest and to calculate the unknown vector of state variables.

The unknown vector which fully determines the volatility structure is in the following denoted ψ.

We find that the measurement equation is given by the static part of the factor structure, as follows:

$$\begin{gathered} R(x,x+k\gamma) = R(0,k\gamma) + Z(\psi)[F_x - F_0] + \varepsilon_x \\ \textit{for} \\ Z(\psi) = Tb(k\gamma)^T \\ \textit{and} \\ \varepsilon_x \approx N(0,\sigma^2 k\gamma I) \end{gathered} \tag{76}$$

As it is seen from equation 76 we have assumed that the individual interest rates are independent, since the covariance matrix is a diagonal matrix. σ^2 is an unknown parameter

[12] The theory behind this section is from Harvey (1992).

that is being estimated together with the process parameters, $k\gamma$ is the time to maturity and I is the identity matrix.

From equation 76 we can also conclude:

- The time heterogenity only comes in effect for $t = 0$.
- The successive yield-curves are then constructed under the time homogenous case

The reason for choosing this procedure is that no knowledge about the state-variables or the parameters that drive the process can be derived from the yield-curve if we allow for time heterogenity, as mentioned in section 7. Time heterogenity would have implied that we would have tried to fit the evolution in the state-variables from time x-1 to x knowing the yield-curve at time x-1 without measurement error - this technique would imply a one-period lagged error correction model.

Furthermore, we know that the transition equation is given as a first order Markov process, as follows:

$$\begin{gathered} F_x = \Phi_x(\psi) F_{x-1} - c_x(\psi) + \eta_x \\ \textit{for} \\ c_x(\psi) = \int_{t_{x-1}}^{t_x} b(t_x - s)^T \lambda S^T ds \\ \Phi_x(\psi) = e^{-D(t_x - t_{x-1})} \\ \eta_x \approx N(0, V_x) \\ V_x = \int_{t_{x-1}}^{t_x} b(t_x - s)^T S S^T b(t_x - s) ds \end{gathered} \tag{77}$$

where the transition equation is the discrete time distribution of the variables. In the Gaussian case, this distribution is the solution to the stochastic differential equation given in the dynamic part of the factor structure[13].

If $t_x - t_{x-1}$ is equal for all x then $c_x(\psi)$, $\Phi_x(\psi)$ and V_x will be time-homogenous. The relations derived here will however be formulated for the general case - ie $t_x - t_{x-1}$ is not equal for all x.

The state space model we set up here is identical in appearance with the Gaussian state space model from Harvey (1992) chapter 3 and with Lund´s (1997) state space model for the Beaglehole and Tenney model.

[13] See for example Harvey (1992) chapter 9.

As can be seen from the measurement equation and the transition equation the model is Gaussian - which mean that we can utilize the linear Kalman filter algorithm. This has the consequence that the estimates of the state variables are optimal in the MSE (mean square error) sense.

In the following, we will call $\hat{F}_{x|x-1}$ and $\hat{F}_x$ respectively the optimal estimator for the vector of state variables based on information up to and at the points in time t_{x-1} and t_x. The optimal estimator is given by the conditional mean of F_x, and will be denoted $E_{x-1}[\circ]$ and $E_x[\circ]$.

The prediction step is given by:

$$\hat{F}_{x|x-1} = E_{x-1}[F_x] = \Phi_x(\psi)\hat{F}_{x-1} - c_x(\psi) \tag{78}$$

which has an MSE matrix identical with:

$$\Sigma_{x|x-1} = E_{x-1}[(F_x - \hat{F}_{x|x-1})(F_x - \hat{F}_{x|x-1})^T] = \Phi_x(\psi)\Sigma_{x-1}\Phi_x^T(\psi) + V_x \tag{79}$$

In the update step the additional information given by $R_x(t,t + k\gamma)$ is used in order to achieve a more precise estimate for F_x, so that:

$$\begin{gathered}
\hat{F}_x = E_x[F_x] = \hat{F}_{x|x-1} + \Sigma_{x|x-1}Z(\psi)^T T_x^{-1} v_x \\
\Sigma_x = \Sigma_{x|x-1} - \Sigma_{x|x-1}Z(\psi)^T T_x^{-1} Z(\psi)\Sigma_{x|x-1} = [\Sigma_{x|x-1}^{-1} + Z(\psi)^T H^{-1} Z(\psi)]^{-1} \\
\textit{for} \\
v_x = R(x,x + k\gamma) - [R(0,k\gamma) + Z(\psi)[\hat{F}_{x|x-1} - F_0]] \\
T_x = Z(\psi)\Sigma_{x|x-1}Z(\psi)^T + H \\
\textit{and} \\
H = \sigma^2 k\gamma I
\end{gathered} \tag{80}$$

where this new estimate of F_x is called the filtered estimate.

The main purpose of the Kalman filter is to extract information about F_x from the observed zero-coupon rates. The Kalman filter algorithm can however also be used to calculate the likelihood function (in our case the exact maximum likelihood function) by using the prediction error decomposition technique.

If we disregard a constant, the log-likelihood function is given by:

$$\log L(R_1,R_2,.....,R_L;\psi) = \sum_{x=2}^{L} \log L(R_x|R_{x-1},.......,R_1;\psi) + \log L(R_1;\psi)$$
$$= -\frac{1}{2}\sum_{x=1}^{L} Log|T_x| - \frac{1}{2}\sum_{x=1}^{L} v_x^T T_x^{-1} v_x \qquad \textbf{(81)}$$

In order to save calculation time, the determinant and the inverse of T_x can be calculated more efficiently using the following expression (see Harvey (1992) page 108):

$$T_x^{-1} = H^{-1} - H^{-1}Z(\psi)\Sigma_x Z(\psi)^T H^{-1}$$
$$and \qquad \textbf{(82)}$$
$$|T_x| = |H|\,|\Sigma_{x|x-1}|\,|\Sigma_x^{-1}|$$

This saves time particularly when L is (much) larger than m, which is generally the case.

In order to start the Kalman filter algorithm we need an initial estimate of the vector of state variables F_0 and its MSE Σ_0 . If the process for the state variable is stationary, its unconditional mean and covariance matrix are used, which can be found, respectively, by letting $t_{t-1} \rightarrow -\infty$ in the expression for $\Phi_x(\psi)$ - $c_x(\psi)$ and V_x in formula 77, see also Harvey (1992) page 121. If, however, some of the state variables are not stationary, the diffuse-prior technique can be used, see Harvey (1992) section 3.4.3.

The unknown parameter vector ψ can now be found by maximising the log-likelihood function from formula 81 by using a suitable algorithm.

With the exception of Bhar and Chiarella (1995 no. 54) this is the first time in the literature that the estimation of the parameters in a multi-factor HJM Markovian structure has been formulated in state space form. The Markovian assumption is the reason why it is possible to estimate the HJM framework by using state space terminology.

The greatest difference between Bhar and Chiarella's approach can briefly be explained as follows: First, they look at the price of a single instrument, and thus their measurement equation is of the form $[1,0,....,0]F_x$, where the Boolean vector in front of F_x works in the same way as our $Z(\psi)$. Second, they estimate the model under the equivalent probability measure Q (risk-neutral probability measure). Third, the procedure here gives a direct expresson of the hedging and risk management problem, as it is clear that the uncertainty is given by $Tb(k\gamma)^T$ – ie the uncertainty is only a function of the k_i´s and therefore (although rather surprisingly) not a function of the σ_i´s.

7.2 Specifying the number of state variables

One final point that has to be established before we can begin to estimate the model using the technique from section 7.1 is the number of state variables and how they are to be specified.

In a number of empirical analyses[14] it has been demonstrated that the dynamics in the term structure in general is driven by 3 factors: a level factor, a slope factor and a curvature factor.

Although it is pointed out in Appendix A that there were certain technical problems connected with the framework adopted in these investigations - that is, how a PCA analysis relates the dynamics in the term structure of interest rates to the individual columns of the factor loadings matrix $b(T\text{-}t)^T$.

Let me dwell on this for a moment: PCA analysis consists of computing the eigenvalues and eigenvectors of the empirical covariance matrix for a given set of maturities using either yields, period returns or forward rates. It is not straightforward to understand why this kind of analysis on different markets using different data, and in all cases find that the first eigenvector (ie, the eigenvector associated with the highest eigenvalue) looks very must like a level-factor. Furthermore this first factor accounts in all cases for around 80-90% of the variation in the yield-curve.

Let us now assume that one of the eigenvalues (κ_i's) is equal to zero (0), this means that the factor associated with this eigenvalue will no longer be stationary (let us assume that $\kappa_1 = 0$). If we perform a PCA analysis on a mixture of stationary and non-stationary components, then the non-stationary component will play the predominant role. As we also find that the factor-loading matrix B will look like: $\begin{pmatrix} 1 & \gamma & e^{-\kappa_2\gamma} & . & e^{-\kappa_m\gamma} \\ 1 & 2\gamma & e^{-\kappa_2 2\gamma} & . & e^{-\kappa_m 2\gamma} \\ . & . & . & . & . \\ 1 & m\gamma & e^{-\kappa_2 m\gamma} & . & e^{-\kappa_m m\gamma} \end{pmatrix}$ - we can conclude:

- A first eigenvalue which is much higher than the others, because the factors do not have the same order of magnitude
- A typical level-factor (the first column in the factor loading matrix B)
- A typical slope-factor (because the second eigenvector will be a linear combination of the first-and second column in the factor loading matrix B)

The existence of a non-stationary factor that is related to the level of yields could therefore be the explanation of the results normally obtained using PCA on historical yield-curve movements.

Despite these shortcomings in using a PCA approach we will assume that three factors with the above-mentioned characteristics are a relevant assumption.

[14] See, among others, Litterman and Scheinkman (1988).

We thus assume that the number of Wiener processes = 3.

The volatility structure for the forward rates for the slope factor is assumed to be of the following form:

$$\sigma^F_{F^s}(t,T) = \sigma_1 e^{-\kappa_1(T-t)} \tag{83}$$

for $0 < \kappa_1$, where it follows that the process for the short factor is defined by a volatility structure à la the Vasicek model (the exponential-decay model).

The volatility structure for the forward rates for the level factor is assumed to be defined as[15]:

$$\sigma^F_{F^l}(t,T) = \sigma_2 e^{-\kappa_2(T-t)} \tag{84}$$

For this factor we allow for the existence of a negative mean-reversion parameter - however bounded to a finite small negative number.

Finally, it is assumed that the volatility structure for the forward rates for the curvature factor is as follows:

$$\sigma^F_{F^v}(t,T) = [a_0 + a_1(T - t)]e^{-\kappa_3(T-t)} \tag{85}$$

where it is assumed that $0 < \kappa_3$, a_0,$a_1 > 0$. Under these assumptions, we find that the left point of intersection with the x-axis (maturity axis) is in origo.

We can thus draw the conclusion that the unknown parameter vector ψ consists of the following 10 elements $\psi = \{\kappa_1, \kappa_2, \kappa_3, \sigma_1, \sigma_2, a_0, a_1, \lambda_1, \lambda_2, \lambda_3\}$.

We thus find that the volatility structure can be written as:

$$\sigma^F(t,T) = \sigma_1 e^{-\kappa_1(T-t)} + \sigma_2 e^{-\kappa_2(T-t)} + [a_0 + a_1(T - t)]e^{-\kappa_3(T-t)} \tag{86}$$

Due to the assumption about the volatility structure for the forward rates for the curvature factor, we know that the dynamic part must be driven by four state variables (as the algebraic multiplicity is > 1), as a volatility structure à la formula 85 can only appear when the k's are not unique (see Appendix A). We thus define yet another state variable F_t^1 which is assumed to have volatility structures as follows:

[15] That is we assume that the level-factor is non-stationary.

$$\sigma^F_{F^1}(t,T) = a_1 e^{-\kappa_3(T - t)} \tag{87}$$

The only point lacking here is a definition of the matrices $b(T-t)^T$ and D.

If we now assume that the vector of state variables F_t is defined as follows:

$$F_t = \begin{pmatrix} F_t^s \\ F_t^l \\ F_t^v \\ F_t^1 \end{pmatrix} \tag{88}$$

then $b(T-t)^T$ can be expressed as:

$$b(T - t)^T = \begin{pmatrix} e^{-\kappa_1\gamma} & e^{-\kappa_2\gamma} & \gamma e^{-\kappa_3\gamma} & e^{-\kappa_3\gamma} \\ e^{-\kappa_1 2\gamma} & e^{-\kappa_2 2\gamma} & 2\gamma e^{-\kappa_3 2\gamma} & e^{-\kappa_3 2\gamma} \\ . & . & . & . \\ e^{-\kappa_1 m\gamma} & e^{-\kappa_2 m\gamma} & m\gamma e^{-\kappa_3 m\gamma} & e^{-\kappa_3 m\gamma} \end{pmatrix}. \quad for\ T - t\ \varepsilon\ [\gamma, 2\gamma, ...,m\gamma] \tag{89}$$

It may also be noted that matrix D is no longer a diagonal matrix, more precisely it is of the following form:

$$D = \begin{pmatrix} e^{-\kappa_1 t} & 0 & 0 & 0 \\ 0 & e^{-\kappa_2 t} & 0 & 0 \\ 0 & 0 & e^{-\kappa_3 t} & e^{-\kappa_3 t} \\ 0 & 0 & 0 & e^{-\kappa_3 t} \end{pmatrix} \tag{90}$$

Finally, we find that $b(t)^T S$ is defined as:

$$b(t)^T S = \begin{pmatrix} \sigma_1 e^{-\kappa_1 t} & 0 & 0. & 0 \\ 0 & \sigma_2 e^{-\kappa_2 t} & 0 & 0 \\ 0 & 0 & P_1(t) e^{-\kappa_3 t} & 0 \\ 0 & 0 & 0 & a_1 e^{-\kappa_3 t} \end{pmatrix} \quad \textbf{(91)}$$

where $P_g(x) = \sum_{i=0}^{g} a_{i+1} x^i$.

This clearly shows that the number of factors (dimension of the vector F_t, dim F_t) has nothing to do with the number of independent Wiener processes (dim dW), apart from the fact that dim $F_t \geq$ dim dW.

It may also be noted that correlation between the individual Wiener processes is achieved by relaxing the assumption that S must be a diagonal matrix[16].

In Brace and Musiela (1997) they use a non-parametric and time dependent volatility structure in order to fit caps and swaption prices.

I, however believe that the parametric approach could be an interesting candidate, because the model specification here shows that it is possible to specify quite general volatility structures, even when we restrict ourselves to Markovian[17] volatility structures.

The Brace and Musielas model is Markovian in the entire term structure of interest rates – ie their model is in principle "infinite"[18] . The model established here can however in the extreme sense also be considered being "infinite" – namely if $k \to \infty$ and all the κ_i's are unique.

The model set up in this section is the one we have chosen to estimate on the Danish bond market[19].

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[16] In the current analysis, we will however assume that the Wiener processes are independent. However, please note that if the k_i 's are not unique, time-dependence is introduced into the covariance matrix, and as a consequence numerical integration has to be carried out in formula 77.

[17] See also Bhar and Chiarella (1995 no. 53).

[18] The Kennedy (1994) model is also a "infinite" model, as it involves the circular Wiener process which is a Wiener process in infinite dimensions.

[19] As pointed out above, the volatility structure set up here is Markovian. However, this is not the only formulation of volatility structures for forward rates which is Markovian, see Appendix B for further information.

8: Estimation results

The analysis period ranges from 2 January 1990 - 30 June 1998.

In practice, it would presumably be most sensible to use a combination of time series and cross-section estimation. This would allow us to use market prices for options, caps and swaptions to provide information about the volatility structure, while at the same time it is possible to estimate the risk parameters – ie a simultaneous determination of the volatility structure and the market price of risk parameters.

The model formulation that has been carried out here therefore makes it possible - in principle - to estimate the volatility structures by extracting information on volatility-based instruments and at the same time determine the market price of risk parameters[20]. It is namely the case that the market price of risk parameters have to be determined before any statement about the dynamics of the underlying securities (bonds etc) can be made. The determination of the market price of risk parameters is thus a requisite for the definition of a term structure model under the original probability measure.

More precisely, the estimation technique shown here can be used for: First, fitting the prices on quoted options. Second, a simultaneous determination of the market price of risk parameters can be carried out. Third, it is possible at the same time to test whether the specified volatility functions can be assumed to be the "correct" ones – this may namely be expected to be the case if it is possible to estimate option prices which lie within the bid-ask spread over a time period with constant volatility parameters. This would be an interesting line of research, which however lies outside the boundaries of this paper.

In table 1 below we show some statistics for the vector of maturity dates we have chosen to estimate our model against:

Table 1: Some Statistics for the Danish Bond Market
(for the period 2 January 1990 - 30 June 1998)

	3-Month Rate	6-Month Rate	1-Year Rate	2-Year Rate	3-Year Rate	4-Year Rate	5-Year Rate	10-Year Rate	12.5-Year Rate
Mean	7,38	7,16	7,04	7,10	7,24	7,37	7,49	7,83	7,91
Variance	51,87	24,38	18,06	14,08	11,04	8,87	7,44	4,72	4,14

[20] However, a technique such as this cannot be tested on the Danish market, as the options market is too illiquid.

Skewnes s	0,19	0,07	0,03	0,01	0,02	0,03	0,04	-0,07	-0,12
Kurtosis	1,73	1,47	1,48	1,55	1,62	1,70	1,79	2,33	2,52
ACF[21] 1	0,99	0,99	0,99	0,99	0,99	0,99	0,99	0,98	0,98
ACF 5	0,95	0,96	0,96	0,96	0,96	0,95	0,95	0,93	0,92
ACF 10	0,90	0,92	0,93	0,92	0,91	0,90	0,89	0,85	0,84
ACF 20	0,84	0,86	0,85	0,83	0,80	0,78	0,76	0,68	0,66
ACF 50	0,61	0,64	0,62	0,56	0,51	0,46	0,41	0,24	0,18

In the table below the estimated parameters are shown. In this connection we might mention that in all cases the different optimization procedures converged to approximately the same results[22]:

Table 2: Parameter Estimates for the 4-factor Markovian model
(period 2 January 1990 - 30 June 1998)

	Parameter Value	Standard-Error[23]
Mean-Reversion - κ_1	0,0281	0,00032
Mean-Reversion - κ_2	0,3438	0,02909
Mean-Reversion - κ_3	3,5036	0,04192
Volatility - σ_1	0,0133	0,00041
Volatility - σ_2	0,0253	0,00132
Volatility - a_0	0,2930	0,02157
Volatility - a_1	0,1768	0,04175
Market Price of Risk - λ_1	-0,0219	0,00407

[21] ACF x - represents the autocorrelation-function for x-lags.

[22] The standard-errors are calculated using the exact Hessian matrix at the optimal parameter estimates. Note that we have used both the BFGS algorithm (see Hald (1979)) and BHHH algorithm in our estimation.

[23] In connection with the standard errors it is of importance to point out the following. They should not be taken as being literal as the big difference between the estimated parameter and the standard error is not only due to efficiency, but also (maybe more) related to the fact that the exact Hessian is not robust in the Newey-West sense - that is we (probably) have a certain degree of autocorrelation and heteroskedasticy. This is also the explanation why we have not included the t-statistics in table 2 as we are led to believe that the statistic is not t-distributed. But despite these short-comings, the parameters appear to be fairly significant.

Market Price of Risk - λ_2	-0,1017	0,05109
Market Price of Risk - λ_3	-0,2487	0,02012

The most striking features of this table are:

- The market price of risk parameter is as expected all negative
- We have that 2 of the state variables exhibit a high degree of mean-reversion
- All the parameters are significant
- We have 1 state variable (state variable no. 1) that is close to being non-stationary, as the mean-reversion is fairly small

Even though our approach is different from the methodology usually used in the literature for multi-factor term structure models a general result is also that one of the state variables has a high degree of mean-reversion (this is usually the spot-rate), see for example Duan and Simonato (1995), Chen and Scott (1995), Andersen and Lund (1997) and Madsen (1998a). It is also well documented in the literature that the level-related process is close to being non-stationary - this is also what we observe.

In table 3 below we show some statistics for the model-generated interest-rate series which are also shown in table 1:

Table 3: Some Statistics for the model-generated interest rate series (for the period 2 January 1990 - 30 June 1998)

	3-Month Rate	6-Month Rate	1-Year Rate	2-Year Rate	3-Year Rate	4-Year Rate	5-Year Rate	10-Year Rate	12.5-Year Rate
Mean	7,35	7,11	6,98	7,05	7,20	7,35	7,46	7,77	7,86
Variance	343,10	71,56	23,57	15,01	12,00	9,67	8,04	5,13	4,65
Skewness	0,74	0,33	0,12	0,03	0,03	0,04	0,04	-0,06	-0,08
Kurtosis	3,56	2,02	1,57	1,57	1,65	1,74	1,84	2,49	2,67
ACF 1	0,97	0,98	0,98	0,98	0,98	0,98	0,97	0,95	0,94
ACF 5	0,89	0,93	0,95	0,95	0,94	0,93	0,93	0,89	0,86
ACF 10	0,82	0,88	0,90	0,90	0,89	0,88	0,87	0,81	0,78
ACF 20	0,73	0,80	0,83	0,82	0,80	0,77	0,75	0,65	0,60
ACF 50	0,46	0,57	0,61	0,56	0,51	0,45	0,40	0,22	0,15

Comparing the results for the model-generated interest rate series from table 3 with the actual interest rate series from table 1 it can be seen that they share the same characteristics. It is however worth pointing out that the variance in the model-generated interest rate series - at least for bonds with short maturities - has a significantly higher variance than in the original/actual interest rate series.

In order to assess the quality of the fit of the model we have performed the following regression:

$$R_t = b\tilde{R}_t + \varepsilon \qquad \textbf{(92)}$$

where R_t is the actual interest rate series, $\tilde{R}_t$ i the model-generated interest rate series, b is the slope and ε is a normally distributed error-term.

As is well known then, we want to estimate a slope that is close to 1 and at the same time have a high degree of correlation (R^2). This is because optimally all the observations should lie on a straight-line with a 45-degree angle.

In table 4 below we have presented the results we get by performing this regression on each of the interest series.

Table 4: Within-Sample Regressions of actual yields contra model-generated yields (for the period 2 January 1990 - 30 June 1998)

	3-Month Rate	6-Month Rate	1-Year Rate	2-Year Rate	3-Year Rate	4-Year Rate	5-Year Rate	10-Year Rate	12.5-Year Rate
Slope	0,9711	0,9915	1,0030	1,0050	1,0032	1,0023	1,0026	1,0075	1,0059
Std. Error	0,0062	0,0043	0,0026	0,0015	0,0013	0,0012	0,0011	0,0012	0,0014
R^2	0,983	0,992	0,997	0,999	0,999	0,999	0,999	0,999	0,999

From this we conclude that there is a very high degree of correlation between the actual interest rate series and the model-generated interest rate series.

8.1 Factor loadings and the effect of the state variables on the shape of the Yield-Curve

In this section we will investigate which kind of factor-loadings the model implies. The factor loadings are given by the matrix $Tb(k\gamma)^T$ (see formula 74), where $b(k\gamma)^T$ is given in formula 89.

The factor loadings are shown below in figure 1:

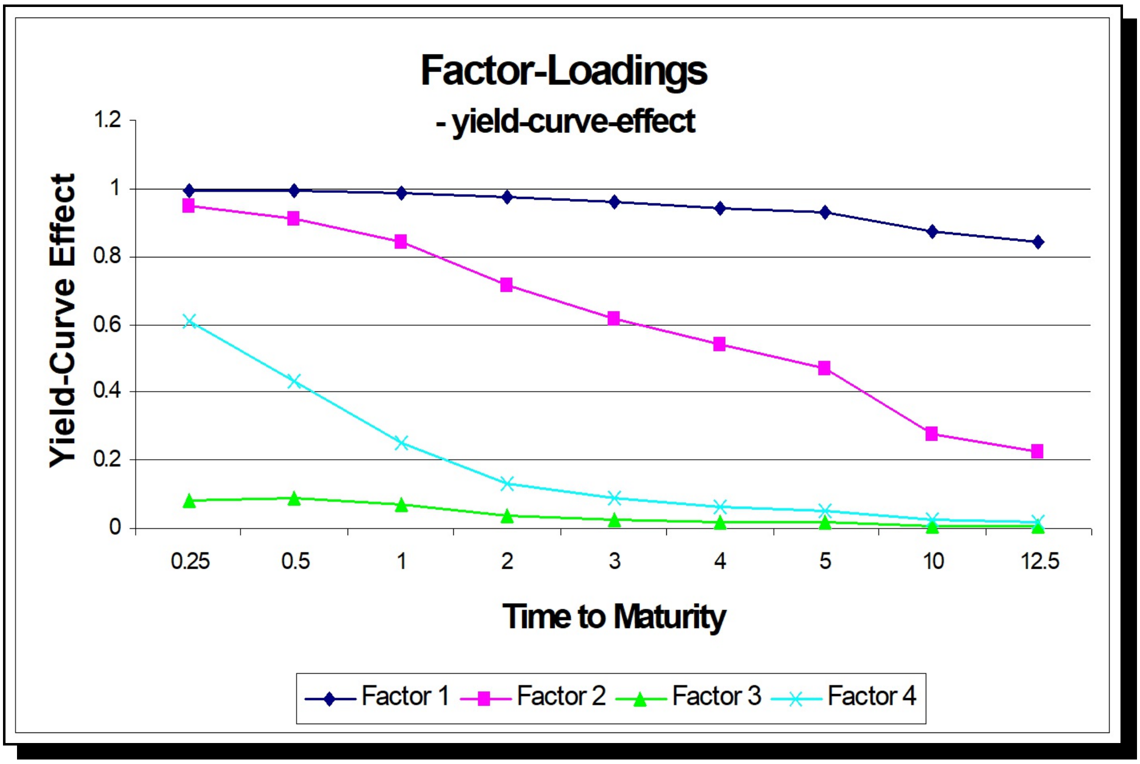

Figure 1

From this figure we can observe the following:

- A factor (factor 1) which appears as the level-factor - as mentioned earlier it is close to being non stationary
- A factor which appears as the slope-factor (factor 2)
- Factor 3 and 4 appears as a slope-factor and a level-factor respectively

It is natural to conclude from this that the curvature in the yield-curve is captured by different signs in the state variables.

At least with respect to factor 1 and 2 our results are in line with the results reported in for example Litterman and Scheinkman (1988) where 3-dominant factors for the evolution were identified using PCA.

We will not try to pursue an economic intepretation here of the 4 state variables, instead we will present a few stylized facts about the evolution in them[24]. In figure 2 below we have shown the evolution in the state variables over the period 2 January 1990 - 30 June 1998.

[24] It is of course an interesting line of research, which for the sake of brevity will be omitted here.

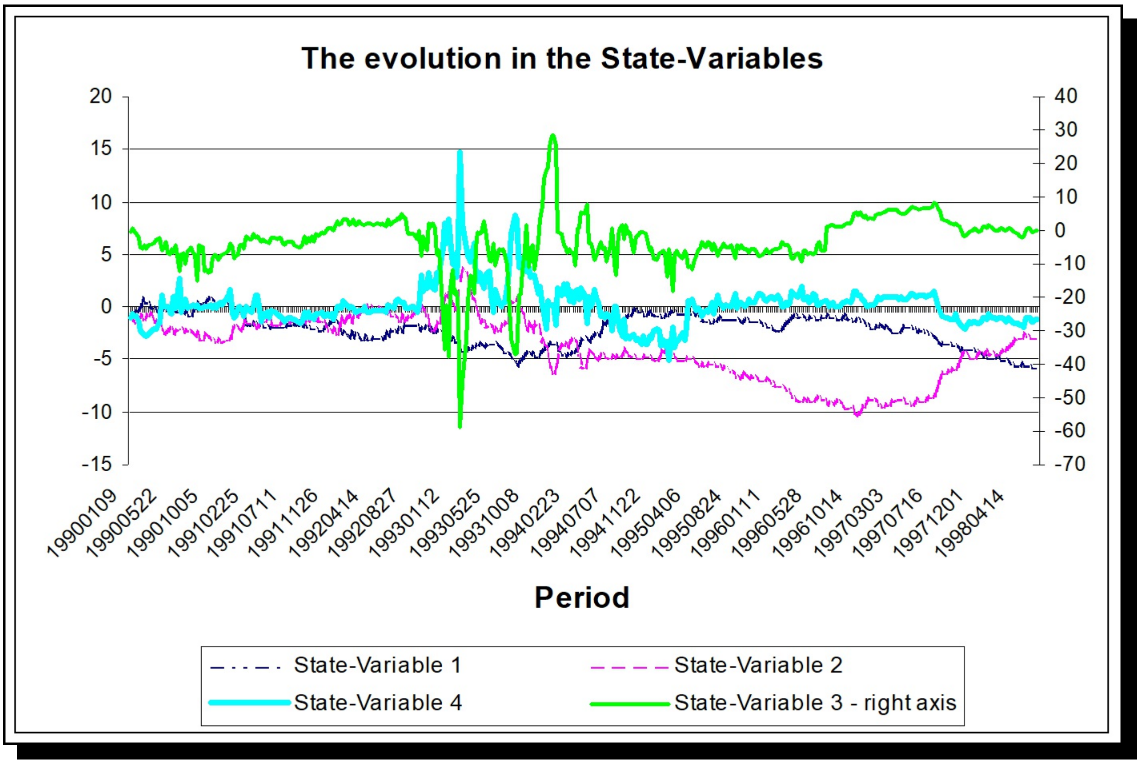

Figure 2

In table 5 below we show various statistics for the state variables:

Table 5: Some Statistics for the State variables
(for the period 2 January 1990 - 30 June 1998)

	State-Variable 1	State-Variable 2	State-Variable 3	State-Variable 4
Mean	-2,21	-4,04	-3,10	0,17
Variance	7,94	99,35	54298,10	179,59
Skewness	-0,493	-0,340	-1,507	1,851
Kurtosis	2,470	2,192	12,727	11,166
ACF 1	0,985	0,991	0,865	0,876
ACF 5	0,906	0,943	0,465	0,699
ACF 10	0,806	0,901	0,285	0,522
ACF 20	0,602	0,839	0,092	0,337
ACF 50	0,030	0,583	-0,216	-0,122

We have the following comments:

- Especially factors 1 and 2 are highly persistent, showing autocorrelations for

10-days at no less than 80%

- For all the factors we observe skewness and kurtosis that imply non-normal shapes
- For especially factors 3 and 4 we see very high third and fourth order moments

In the table 6 below we have calculated the correlations matrix for the 4 state variables:

Table 6: Standardised Correlations Matrix for the State variables
(for the period 2 January 1990 - 30 June 1998)

	State-Variable 1	State-Variable 2	State-Variable 3	State-Variable 4
State-Variable 1	1,00	-0,21	-0,05	-0,30
State-Variable 2	-0,21	1,00	-0,40	0,18
State-Variable 3	-0,05	-0,40	1,00	-0,52
State-Variable 4	-0,30	0,18	-0,52	1,00

We can from table 6 see that either we have a very low correlation (close to zero (0)) between the state variables, or we have a negative correlation between the state variables. Which is positive from a modelling point of view[25].

The final analysis we will perform is an investigation of the rankings of the 4 state variables. In this connection it is logical to expect that the factor loadings from figure 1 will give us the relative rankings of importance. But some interesting information can nevertheless be derived from the following analysis.

In order to analyse that we have performed the following regression:

$$R_t = a + b_1F_t(1) + b_2F_t(2) + b_3F_t(3) + b_4F_t(4) + \varepsilon \qquad \textbf{(93)}$$

where R_t is the actual interest rate series, a is the interception with the y-axis, b_i, for i = [1,2,3,4], is the slope, $F_t(i)$, for i = [1,2,3,4] is the state variables and ε is a normally distributed error-term.

The results are shown below in table 7:

Table 7: Within-Sample Regressions of actual yields contra State variables
(for the period 2 January 1990 - 30 June 1998)

[25] A very high positive correlation between some of the state variables could have indicated the possible redundance of one of the state variables.

	3-Month Rate	6-Month Rate	1-Year Rate	2-Year Rate	3-Year Rate	4-Year Rate	5-Year Rate	10-Year Rate	12.5-Year Rate
Interception	13,57	13,24	12,77	12,28	11,96	11,71	11,51	10,83	10,60
Std. Err.	0,0000	0,0006	0,0023	0,0023	0,0038	0,0040	0,0042	0,0145	0,0226
Factor 1 Slope	0,9957	0,9932	0,9818	0,9749	0,9644	0,9494	0,9327	0,8566	0,8271
Factor 1 Std. Err.	0,0000	0,0001	0,0006	0,0006	0,0010	0,0010	0,0011	0,0036	0,0057
Factor 2 Slope	0,9500	0,9114	0,8370	0,7190	0,6220	0,5409	0,4736	0,2685	0,2097
Factor 2 Std. Err.	0,0000	0,0001	0,0003	0,0003	0,0005	0,0005	0,0005	0,0019	0,0029
Factor 3 Slope	0,0809	0,0893	0,0699	0,0418	0,0279	0,0203	0,0158	0,0073	0,0058
Factor 3 Std. Err.	0,0000	0,0000	0,0001	0,0001	0,0002	0,0002	0,0002	0,0008	0,0013
Factor 4 Slope	0,6096	0,4312	0,2551	0,1276	0,0838	0,0656	0,0567	0,0367	0,0264
Factor 4 Std. Err.	0,0000	0,0001	0,0005	0,0005	0,0008	0,0009	0,0009	0,0031	0,0048
R^2	100,00	100,00	100,00	100,00	99,98	99,98	99,97	99,44	98,46

As can be seen from this table we have that the slope coefficients estimated for each of the state variables for each of the maturities is approximately equal to the appropriate vector of factor loadings.

Furthermore an interesting observation is that the intercepts are equal to the actual interest rates for the initial yield-curve, that is from 2 January 1990. This is completely in line with our comments in connection with formula 76 in section 7.1.

We have the following ranking of importance - which can be derived from either table 7 or figure 1:

- Factor 1 - the level-factor dominates over the other factors
- Factor 2 - the slope-factor is the second most important factor
- For factors 3 and 4 we observe that factor 4 dominates over factor 3 for the maturity dates we analyse but for longer maturites (above 20-years approximately) then factor 3 start to dominate over factor 4

Table 7 also indicates that the 4 state variables are very capable of capturing the evolution through time of the term structure of interest rates - at least within the sample.

8. Conclusion

In a number of empirical studies of the dynamics of the yield-curve, a general conclusion is that 3 factors are necessary (and sufficient) for describing the dynamics of the yield-curve, see among others Litterman and Scheinkman (1988).

These 3 factors are usually being estimated using Principal Component Analysis (PCA) and are further identified as a level factor, a slope factor and a curvature factor.

These studies show that the estimated factor-loadings are of both practical and theoretical relevance as they appear for (as yet) inexplicable reasons to be remarkably stable over time.

In this paper we extended the traditional approach in the literature using PCA by taking into account the dynamic part of the process. In PCA the resulting factor-pattern is namely neither directly nor indirectly analysed.

Using a Gaussian framework we established the relationship between the statistical and dynamic aspects of the linear factor structure taking a probability-based framework as our starting point. In this connection we showed that the identification of the state variables gave rise to a Markovian stochastic system. This Markovian structure was set up by first assuming that the dynamics in the state variables followed a generalised Vasicek model (a multi-dimensional Hull and White) and second that the dynamics in the forward rates was defined in an m-factor model in the HJM framework.

This implied that the relation between the stochastic elements imposed restrictions on the mean-reversion matrix for the state variables, more precisely the mean-reversion matrix resulted in a Frobenius matrix.

We also present a new approach for estimating the dynamic factor structure using the state space terminology combined with the linear Kalman filter.

In connection with the estimation of the dynamic factor structure we noticed the following:

- In the case of time homogenity in the process for the state variables, the observable yield-curve provide us with information about the volatility structure
- In the case of the time heterogenity case in the process for the state variables, we deduced that the observable yield-curve did not provide us with any information about the other parameters that govern the state variables, for example the volatility parameter is in this case not observable.

This observation entails estimation problems. We handled this by only letting the time heterogenity come into effect at time t = 0, and then letting the successive yield-curves be derived under the assumption of time homogenity.

We specified a 4 factor model, that was governed by only 3 Brownian motions. The 4 factor model implied by our definition of the volatility structure was specified in accordance with the results from example Littermann and Scheinkman (1988).

The model was estimated over the period 2 January 1990 - 30 June 1998, and from the results we concluded the following:

- All the parameters are significant
- The market prices of risk parameters were all negative
- We have 1 state variable (state variable no. 1) that is close to being non-stationary, as the mean-reversion is fairly small. This also happened to be the dominant factor
- The second dominant factor was the slope-factor
- The curvature in the yield-curve is captured through the combined effect of the factor loading and the signs and magnitudes of the state variables
- Performing a multiple regression of the state variables on the actual interest rate evolution (in sample) resulted in a R^2 of no less than 98% for all the 9 maturities we analysed

We believe that the analysis both on the theoretical level and on the empirical level gives new insight into the dynamic in the yield-curve.

More research is however required. For example we need to address the identification of the state variables. This is possible, at least from a theoretical point of view, but from a practical implementation point of view - as our results also showed - this might not be so straightforward. It would also be interesting to consider an alternative specification of the volatility structure than the one considered here. As shown in this paper there are fairly rich possibilities of constructing the volatility structure, even in a Markovian structure. For this reason I believe it is worth continuing to pursue the Markovian structure. Another line of research could be to employ this framework on other markets than the Danish, so that we could include options in the estimation procedure, instead of “only” relying on the evolution in the yield-curve.

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Appendix A

Let us consider a matrix M which is given by either formula 55 or formula 59 in the main text and assume that M's eigenvalues are real numbers and specify the characteristic polynomial for M as:

$$\begin{aligned} P(m) = Det[M - aI] &= (-1)^m (a - r_1)^{x_1} (a - r_2)^{x_2} ....(a - r_m)^{x_m} \\ &= \prod_{i=1}^{m} (a - r_i)^{x_i} \end{aligned} \quad \textbf{(94)}$$

for x_i being a positive integer such that $\sum_{i=1} tromx_i = m$, and where x_i is the algebraic multiplicity for the i´th eigenvalue r_i.

From this we can deduce that P(m) is the m'th order polynomial of the following form :

$$P(m) = (-1)^m (\alpha_1 + \alpha_2 a + ... + \alpha_{m-1} a^{m-1} + a^m) \quad \textbf{(95)}$$

where its zeros are the eigenvalues of M.

We have that any matrix of dimension m x m which has m unique eigenvalues can be diagonalised (Theorem 6.2.9 in Stoer and Bulirsch (1993)).

Futhermore, if the matrix is a Frobenius, then Stoer and Bulirsch (1993, Theorem 6.3.4) inform us that the matrix is "non-derogatory". Here it is shown that, if a matrix is "non-derogatory", the minimal polynomial of the matrix is identical to its characteristic polynomial (up to a multiplicative constant), see Stoer and Bulirsch (1993, Corollary 6.2.15.).

By minimal polynomial we mean the polynomial:

$$P(m) = \alpha_1 + \alpha_2 a + ... + \alpha_{m-1} a^{m-1} + a^m \quad \textbf{(96)}$$

of smallest degree having the property $P(M) = 0$.

If M is defined as in formula 55, we will now denote the roots in this polynomial by $-\kappa_1, \kappa_2, ..., \kappa_m$. Alternatively, if M is defined as in formula 59 we will denote the roots in this polynomial $e^{-\kappa_1}, e^{-\kappa_2}, ..., e^{-\kappa_m}$.

In this connection, we will assume that κ_i for i = {1,2,..,m} is a real number larger than zero and less than 1. One consequence of this assumption is that the process for all state variables will be stationary as no stationariness appears for $\kappa \leq 0$.

From this assumption it follows that $b(T\text{-}t)^T$ from formula 46 can be written as:

$$b(T-t)^T = \begin{pmatrix} e^{-\kappa_1\gamma} & e^{-\kappa_2\gamma} & . & . & e^{-\kappa_m\gamma} \\ e^{-\kappa_1 2\gamma} & e^{-\kappa_2 2\gamma} & . & . & e^{-\kappa_m 2\gamma} \\ . & . & . & . & . \\ e^{-\kappa_1 m\gamma} & e^{-\kappa_2 m\gamma} & . & . & e^{-\kappa_m m\gamma} \end{pmatrix} \quad for\ T-t\ \varepsilon\ [\gamma,2\gamma,.......,m\gamma] \qquad \textbf{(97)}$$

when all the κ_i's are unique.

Where the number of rows is equal to the length of the vector (T-t) and the number of rows is identical to the dimension of the Wiener process.

The volatility structure for the forward rates is of the following form

$\sigma^F(t,T) = \sum_{i=1}^{m} issigma_i e^{-\kappa_i(T-t)}$ when all the k_i´s are unique.

Let us now consider another extreme case - namely the case where all the κ_i´s are equal. If this is the case the volatility structures for the forward rates $\sigma^F(t,T)$ from formula 36 are defined as:

$$\sigma^F(t,T) = P_m(T-t)e^{-\kappa_i(T-t)} \qquad \textbf{(98)}$$

where $P_m(T\text{-} t)$ is a polynomial of order g_m - 1, such that:

$$P_m(x) = \sum_{m=0}^{g_m-1} \sigma_m x^{mi} \qquad \textbf{(99)}$$

In this case, the factor loading matrix $b(T\text{-}t)^T$ from formula 97 will be defined as:

$$b(T-t)^T = \begin{pmatrix} e^{-\kappa\gamma} & \gamma e^{-\kappa\gamma} & . & . & \gamma^{m-1}e^{-\kappa\gamma} \\ e^{-\kappa 2\gamma} & 2\gamma e^{-\kappa 2\gamma} & . & . & (2\gamma)^{m-1}e^{-\kappa 2\gamma} \\ . & . & . & . & . \\ e^{-\kappa m\gamma} & m\gamma e^{-\kappa m\gamma} & . & . & (m\gamma)^{m-1}e^{-\kappa m\gamma} \end{pmatrix} \quad for\ T-t\ \varepsilon\ [\gamma,2\gamma,.......,m\gamma] \quad \textbf{(100)}$$

A volatility structure identical to the relationship from formula 98 has independently been derived by Bhar and Chiarella (1995 no. 53) in a transformation of HJM models to Markovian structures.

We will now take a closer look at the conditions that must be fulfilled so that the state variables can be identified with particular interest rates.

Let us therefore define a selection matrix U, consisting of zeros and ones, so that:

$$U^T R = Z_{tZ_t} = R \tag{101}$$

wher Z_t is a vector of state variables and R is a vector of interest rates. We want $U^T Z_{tR}$ to be identifiable with one or more of the state variables from Z_t. In general we have that the dimension of $Z_t \leq$ the dimension of R[26].

Thus what we want to find is a matrix T which is invertible, so that:

$$TZ_t = U^T R + deterministic\ part \tag{102}$$

As formula 32 in the main text tells us that - it can be seen that the relation from formula 8 is equivalent to $T = U^T b^T$, . From this we can deduce that if the factors are to be identified with particular interest rates through the selection matrix U, then $U^T b(^T$ must be invertible.

If $U^T b^T$ is invertible then we have the following relationship:

$$R = b^T (U^T b^T)^{-1} (U^T b^T) Z_t + a \tag{103}$$

That $U^T b(^T$ is invertible follows as a direct consequence of formula 97.

From this we can deduce that the factor structure $R = b^T Z_t + a$ is valid for any linear transformation af Z_t, more precisely, for any invertible matrix K we have:

$$\begin{gathered} R = \tilde{b}^T \tilde{Z}_t + a \\ for \\ \tilde{b}^T = b^T K \\ and \\ \tilde{Z}_t = K^{-1} Z_t \end{gathered} \tag{104}$$

As a consequence, only the space spanned by the columns in b^T can be identified. This means that the factor loadings which can be attributed to the first factor found in a PCA cannot be identified with the first column of the matrix b^T.

[26] R could for example be defined as $r^F(t,T)$ in equation 33.

Furthermore, we find that the number of factors has nothing to do with the number of independent Wiener processes, except that there have to be at least as many factors as there are independent Wiener processes.

Appendix B

Let us now look at a one-factor model. Formula 16 in the main text tells us that the instantaneous spot rate r(t) satisfies the following stochastic integral equation:

$$r(t) = r^F(0,t) + \int_0^t \sigma^F(s,t)\sigma_P(s,t)ds + \int_0^t \sigma^F(s,t;i)d\tilde{W}_i(s) \qquad \textbf{(105)}$$

Alternatively, formula 105 can be expressed by the following stochastic process:

$$dr(t) = \left[r_t^F(0,t) + \partial_t \int_0^t \sigma^F(s,t)\sigma_P(s,t)ds + \int_0^t \partial_t \sigma^F(s,t)d\tilde{W}(s) \right] dt + \int_0^t \sigma^F(s,t)d\tilde{W}(ts) \qquad \textbf{(106)}$$

where it is the third part of the drift specification in formula 106 that in general formulations of the volatility structure means that HJM models are non Markovian.

Let us for now assume that $\sigma^F(t,T) = \sigma e^{-\kappa(T - t)} \Rightarrow \sigma_P(t,T) = \frac{\sigma}{\kappa}[1 - e^{-\kappa(T - t)}]$, where this is the exponential decay model[27]. Under this assumption we can rewrite formula 105 as:

$$r(t) = r^F(0,t) + \int_0^t \frac{\sigma^2}{\kappa} e^{-\kappa(t - s)}[1 - e^{-\kappa(t - s)}]ds + \int_0^t \sigma e^{-\kappa(t - s)}d\tilde{W}(s) \qquad \textbf{(107)}$$

Let us now consider the result from equation 106. The first term in the drift specification follows directly, whereas the second term can be written as:

$$\left[\int_0^t [\sigma_P(s,t)\sigma_t^F(s,t) + \sigma^F(s,t)^2]ds \right] dt = -\left[\int_0^t \sigma^2 e^{-\kappa(t - s)}[1 - 2e^{-\kappa(t - s)}]ds \right] dt \qquad \textbf{(108)}$$

We have that the third term in the drift specification in formula 106 is defined as:

$$\left[\int_0^t \sigma_t^F(s,t)d\tilde{W}(s) \right] dt = -\left[\int_0^t \sigma\kappa e^{-\kappa(t - s)}d\tilde{W}(s) \right] dt \qquad \textbf{(109)}$$

which, given formula 107, can be rewritten as:

[27] Which in structure is identical with the Hull and White model with time-dependent drift parameter.

$$-\left[\int_0^t \sigma\kappa e^{-\kappa(t-s)} d\tilde{W}(s)\right] dt = \left[\kappa r^F(0,t) - \kappa r(t) + \int_0^t \sigma^2 e^{-\kappa(t-s)}[1 - e^{-\kappa(t-s)}] ds\right] dt \qquad \textbf{(110)}$$

From this it follows that the process dr(t) from formula 106, taking into account the assumed volatility structures can be expressed as:

$$dr(t) = \left[r_t^F(0,t) + \kappa[r^F(0,t) - r(t)] + \int_0^t \sigma^2 e^{-2\kappa(t-s)} ds\right] dt + \sigma d\tilde{W}(t) \qquad \textbf{(111)}$$

where this expression is found to be identical with formula 52 in the main text for $\kappa < 0$ and it is also identical with formula 14 in Bhar and Chiarella (1995 no. 53).

Here the well-known result that the Hull and White model is a Markovian model in the short rates is clearly shown by formula 111[28].

Let us now instead look at the following more general formulation for the forward rate volatility structure:

$$\sigma^F(t,T) = \sigma(t,T)G(r)(t)) \qquad \textbf{(112)}$$

Assume now that $\sigma(t,T) = \sigma e^{-\kappa(T-t)}$ and $G(r(t)) = r(t)^v \;\; \mathit{for}$ for $v \geq 0$. We find that for $v = 0$ the form is identical to the volatility structure in the exponential decay model.

With these assumptions in mind, it is easily seen that the forward rates generally can be written as:

$$r^F(t,T) = r^F(0,T) + \int_0^t r(s)^{2v} \sigma^F(s,T) \sigma_P(s,T) ds + \int_0^t r(s)^v \sigma^F(s,T) d\tilde{W}(s) \qquad \textbf{(113)}$$

If we now take the spot rate process as our starting point, we find:

$$r(t) = r^F(0,t) + \int_0^t r(s)^{2v} \sigma^F(s,t) \sigma_P(s,tT) ds + \int_0^t r(s)^v \sigma^F(s,tT) d\tilde{W}(s) \qquad \textbf{(114)}$$

By applying the principles above, we can show that the stochastic process for the spot rate can be expressed as (here shown in abstract form):

[28] An alternative proof that precisely this volatility structure fulfills the Markov characteristic has been carried out in Caverhill (1994).

$$dr(t) = \left[r_t^F(0,t) + \partial_t \int_0^t G(r(s))^2 \sigma(s,t) \int_u^t \sigma(s,u) dy ds + \int_0^t G(r(t)) \partial_t \sigma(s,t) d\tilde{W}(s) \right] dt + G(r(t)) \sigma(t,t) d\tilde{W}(t) \tag{115}$$

On the basis of our assumptions on the functional forms of G(r(t)) and σ(t,T) respectively, we find that formula 115 can be written as:

$$dr(t) = \left[r_t^F(0,t) + \kappa[r^F(0,t) - r(t)] + \sigma^2 \int_0^t r(s)^{2v} e^{-2\kappa(t - s)} ds \right] dt + \sigma r(t)^v d\tilde{W}(t) \tag{116}$$

where this happens to be actually identical with formula 3.4a in Ritchken and Sankarasubramanian (1995).

Let us rewrite formula 116 as:

$$\begin{gathered} dr(t) = \left[r_t^F(0,t) + \kappa[r^F(0,t) - r(t)] + \sigma^2 \varphi(t) \right] dt + \sigma r(t)^v d\tilde{W}(t) \\ for \\ d\varphi(t) = [r(t)^{2v} - 2 lakappa \varphi(t)] dt \end{gathered} \tag{117}$$

where this is the two-dimensional Markov model developed by Ritchken and Sankarasubramanian.

More general model formulations can of course also be made, for example by assuming that σ(t,T) is defined by $\sigma(t,T) = P_i(T - t) e^{-\kappa(T - t)}$ for $P_i(x) = \sum_{i=0}^{g_i - 1} \sigma_t x^i$ where g_i - 1 is the polynomial order. This is treated in more detail in Bhar and Chiarella (1995 no.53).

We have the following relationship between the price P(t,T) and the forward rates:

$$P(t,T) = \tilde{E}\left[\exp\left(-\int_t^T r^F(t,s) ds \right) \right] \tag{118}$$

In the following we will show that yield-curve at any time t is fully specified by the initial yield- curve, the spot rate and φ(t). From formula 9 we have the following relationship:

$$r^F(t,T) = r^F(0,T) + \int_0^t r(s)^{2v} \frac{\sigma^2}{\kappa} e^{-\kappa(T - s)} [1 - e^{-\kappa(T - s)}] ds + \sigma e^{-\kappa T} \int_0^t r(s)^v e^{\kappa s} d\tilde{W}(s) \tag{119}$$

From formula 114 we have:

$$\sigma e^{-\kappa t}\int_0^t r(s)^v e^{\kappa s} d\tilde{W}(s) = r(t) - r^F(0,t) - \int_0^t r(s)^{2v}\frac{\sigma^2}{\kappa}e^{-\kappa(t-s)}[1 - e^{-\kappa(t-s)}]ds \quad \textbf{(120)}$$

such that formula 115 can be expressd as::

$$\begin{aligned} r^F(t,T) = r^F(0,T) &+ \int_0^t r(s)^{2v}\frac{\sigma^2}{\kappa}e^{-\kappa(T-s)}[1 - e^{-\kappa(T-s)}]ds + e^{-\kappa(T-t)}[r(t) - r^F(0,t)] \\ &- e^{-\kappa(T-t)}\int_0^t r(s)^{2v}\frac{\sigma^2}{\kappa}e^{-\kappa(t-s)}[1 - e^{-\kappa(t-s)}]ds \end{aligned} \quad \textbf{(121)}$$

which can be simplified as follows:

$$r^F(t,T) = r^F(0,T) + e^{-\kappa(T-t)}[r(t) - r^F(0,t)] + \frac{\sigma^2}{\kappa}[e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}]\varphi(t) \quad \textbf{(122)}$$

The price expression for P(t,T) now follows directly by inserting formula 122 into formula 118 and carrying out the integration. Formula 122 shows that the bond price (given σ and κ) is completely defined by the initial term structure and the two state variables r(t) og φ(t).

Appendix C

In this appendix we will discuss two methods for estimating the parameters in the HJM framework. However, we will only focus on the estimation of the volatility parameters taking into account the historical development of the term structure.

The first method comes from Heath, Jarrow and Morton (1990). It can be described briefly as follows:

Under the original probability measure P, we find that the process for the forward rates can be written as:

$$\begin{gathered} r^F(t,T) = r^F(0,T) + \mu^F(t,T) + \sum_{i=1}^{m} \int_0^t \sigma^F(s,T;i) dW_i(s) \\ for \\ \mu^F(t,T) = \sum_{i=1}^{m} \int_0^t \sigma^F(s,T;i)\sigma_P(s,T;i) ds - \sum_{i=1}^{m} \int_0^t \sigma^F(s,T;i)\lambda_i ds \end{gathered} \tag{123}$$

From formula 14 in the main text we have that the process that drives the yield-curve is of the following form:

$$\begin{gathered} R(t,T) = R^F(0,t,T) + \mu^R(t,T) + \sum_{i=1}^{m} \int_0^t \frac{\sigma_P(s,T;i)}{T-t} dW_i(s) \\ for \\ \mu^R(t,T) = \sum_{i=1}^{m} \int_0^t \frac{1}{2} \frac{\sigma_P^2(s,T;i)}{T-t} ds - \sum_{i=1}^{m} \int_0^t \frac{\sigma_P(s,T;i)}{T-t} \lambda_i ds \end{gathered} \tag{124}$$

The stochastic differential equation for R(t,T) can be found by applying Ito´s lemma, which gives:

$$dR(t,T) = [R^F(0,t,T) + \mu^R(t,T)]dt + \sum_{i=1}^{m} \frac{\sigma_P(ts,T;i)}{T-t} dW_i(t) \tag{125}$$

With regard to the fact that the volatility structure is only a function of the remaining time to maturity t = T – t and the market price of risk parameters are time homogeneous, we find that the only time-dependent component in the drift from formula 3 arises from the spot rate. To be more precise, the time-dependent component consists of the difference between the instantaneous rate r(t) and $R^F(0,t,T)(T - t)$, so that the drift in the formula can be written as:

$$R^F(0,t,T) + \mu^R(t,T) = \frac{1}{\tau}\left[\sum_{i=1}^{m}\left[\frac{1}{2}\sigma_P^2(0,T-t;i) - \sigma_P(0,T-t;i)\lambda_i\right] + [R^F(0,t,T)\tau - r(t)]\right] \quad \textbf{(126)}$$

Since the term structure can only be estimated at discrete time intervals, it is necessary to discretizise the SDE for the term structure in formula 3, taking into account the drift specification in formula 4, this yields[29]:

$$\begin{aligned}\Delta R(t,T) = {} & \frac{1}{\tau}\left[\sum_{i=1}^{m}\left[\frac{1}{2}\sigma_P^2(0,T-ts;i) - \sigma_P(0,T-t;i)\lambda_i\right] + [R^F(0,t,T)\tau - r(t)]\right]\Delta t \\ & + \sum_{i=1}^{m}\frac{\sigma_P(0,T-t;i)}{tasu}\Delta W_i(t)\end{aligned} \quad \textbf{(127)}$$

Where it is assumed that T - t ε [T_1, T_2,.....,T_n], that is T - t is a vector of n-elements. Furthermore it is assumed that the term structure is observed at L points in time, that is t = [t_1,t_2,...,t_L].

The estimation problem can now be set up in the following form:

$$\begin{pmatrix} \Delta\tilde{R}(t_1,T_1) & \Delta\tilde{R}(t_2,T_1) & . \; . & \Delta\tilde{R}(t_L,T_1) \\ \Delta\tilde{R}(t_1,T_2) & \Delta\tilde{R}(t_2,T_2) & . \; . & \Delta\tilde{R}(t_L,T_2) \\ . & . & . \; . & . \\ \Delta\tilde{R}(t_1,T_n) & \Delta\tilde{R}(t_2,T_n) & . \; . & \Delta\tilde{R}(t_L,T_n) \end{pmatrix}_{|T\,x\,L} = \begin{pmatrix} \frac{\sigma_P(0,T_1;1)}{T_1} & \frac{\sigma_P(0,T_1;2)}{T_1} & . \; . & \frac{\sigma_P(0,T_1;m)}{T_1} \\ \frac{\sigma_P(0,T_2;1)}{T_2} & \frac{\sigma_P(0,T_2;2)}{T_2} & . \; . & \frac{\sigma_P(0,T_2;m)}{T_2} \\ . & . & . \; . & . \\ \frac{\sigma_P(0,T_n;1)}{T_n} & \frac{\sigma_P(0,T_n;2)}{T_n} & . \; . & \frac{\sigma_P(0,T_n;m)}{T_n} \end{pmatrix}_{|T\,x\,m}$$
$$\times \begin{pmatrix} \Delta W_1(t_1) & \Delta W_1(t_2) & . \; . & \Delta W_1(t_L) \\ \Delta W_2(t_1) & \Delta W_2(t_2) & . \; . & \Delta W_2(t_L) \\ . & . & . \; . & . \\ \Delta W_m(t_1) & \Delta W_m(t_2) & . \; . & \Delta W_m(t_L) \end{pmatrix}_{|m\,x\,L} \quad \textbf{(128)}$$

where the foot signs give the dimension of the individual matrices. This expression is seen to be identical with HJM's (1990) formula 19, except that theirs is based on the forward rates.

We have that:

[29] Which can be recognized as being an Euler discretization, see Kloeden and Platen (1995).

$$\Delta\tilde{R}(t,T) = \Delta R(t,T)$$
$$-\frac{1}{\tau}\left[\sum_{i=1}^{m}\left[\frac{1}{2}\sigma_P^2(0,T-t;i) - \sigma_P(0,T-t;i)\lambda_i\right] + [R^F(0,t,T)\tau - r(t)]\right]\Delta t \qquad \textbf{(129)}$$

Given this expression, it is possible to use a principal component analysis to estimate the volatility structure, ie a PCA can be employed to determine the volatility structure in a non parametrical HJM model.

The method can however only be employed if the volatility structure is of the following form[30]:

$$\sigma_P(T-t) = \sigma(T-t) \qquad \textbf{(130)}$$

ie volatility structures which are only a function of the remaining time to maturity. It is thus not possible to specify/estimate the mean-reversion parameter κ. Because of that mean-reversion can only be determined indirectly, where mean-reversion can be observed if the volatility structure declines with the rise in T - t.

An alternatively estimation method can be derived by observing that the actual change in the yield-curve adjusted with $\frac{1}{\tau}[R^F(0,t,T)\tau - r(t)]$ is normally distributed. More precisely we have the following relationship:

$$\Delta R(t,T) - \frac{1}{\tau}[R^F(0,t,T)\tau - r(t)]\Delta t$$
$$\approx N\left(\sum_{i=1}^{m}\left[\frac{1}{2}\sigma_P^2(0,T-t;i) - \sigma_P(0,T-t;i)\lambda_i\right]\Delta t, \frac{1}{\tau}\sum_{i=1}^{m}\sigma_P^2(0,T-t;i)\Delta t\right) \qquad \textbf{(131)}$$

Taking this statement as our starting point, it is possible to estimate the vector of parameters that describe the volatility structure. This estimation method is however - in contrast with the PCA based-method - parametric.

If we now regard a time series of L yield-curve changes for maturities defined by the vector T-t, then we can estimate the unknown parameter vector using the following relation:

$$\frac{Var(T-t)}{\Delta t} = \frac{1}{\tau}\sigma_P^2(0,T-t) \qquad \textbf{(132)}$$

[30] It is however also possible to incorporate the characteristic that the forward rate process does not contain negative interest rates. For a more detailed description refer to HJM (1990).

where Var(T - t) represents the sampling variance for the changes in the yield-curve.

It has been pointed out by Heitmann and Trautmann (1995) that an estimation of the unknown parameter vector using equation 9 by minimising the squared residuals is equivalent to using the GMM method based on only the second moment and with a weighting matrix identical to the identity matrix.

Finally, it is worth noting that the GMM method has been used by Jeffrey (1995) and Au, Sim and Thurston (1997) for testing various Markovian parametric specifications of the volatility structure. Please refer to these papers for further information.

Appendix D

Let us now look at a one-factor model. Formula 16 in the main text tells us that the instantaneous spot rate r(t) satisfies the following stochastic integral equation:

$$r(t) = r^F(0,t) + \int_0^t \sigma^F(s,t)\sigma_P(s,t)ds + \int_0^t \sigma^F(s,t;i)d\tilde{W}_i(s) \qquad \textbf{(133)}$$

Alternatively, formula 1 can be expressed by the following stochastic process:

$$dr(t) = \left[r_t^F(0,t) + \partial_t \int_0^t \sigma^F(s,t)\sigma_P(s,t)ds + \int_0^t \partial_t \sigma^F(s,t)d\tilde{W}(s) \right] dt + \int_0^t \sigma^F(s,t)d\tilde{W}(ts) \qquad \textbf{(134)}$$

where it is the third part of the drift specification in formula 2 that in general formulations of the volatility structure means that HJM models are non Markovian.

Let us for now assume that $\sigma^F(t,T) = \sigma e^{-\kappa(T - t)} \Rightarrow \sigma_P(t,T) = \frac{\sigma}{\kappa}[1 - e^{-\kappa(T - t)}]$, where this is the exponential decay model[31]. Under this assumption we can rewrite formula 1 as:

$$r(t) = r^F(0,t) + \int_0^t \frac{\sigma^2}{\kappa} e^{-\kappa(t - s)}[1 - e^{-\kappa(t - s)}]ds + \int_0^t \sigma e^{-\kappa(t - s)} d\tilde{W}(s) \qquad \textbf{(135)}$$

Let us now consider the result from equation 2. The first term in the drift specification follows directly, whereas the second term can be written as:

$$\left[\int_0^t [\sigma_P(s,t)\sigma_t^F(s,t) + \sigma^F(s,t)^2]ds \right] dt = -\left[\int_0^t \sigma^2 e^{-\kappa(t - s)}[1 - 2e^{-\kappa(t - s)}]ds \right] dt \qquad \textbf{(136)}$$

We have that the third term in the drift specification in formula 2 is defined as:

$$\left[\int_0^t \sigma_t^F(s,t)d\tilde{W}(s) \right] dt = -\left[\int_0^t \sigma\kappa e^{-\kappa(t - s)} d\tilde{W}(s) \right] dt \qquad \textbf{(137)}$$

which, given formula 3, can be rewritten as:

[31] Which in structure is identical with the Hull and White model with time-dependent drift parameter.

$$-\left[\int_0^t \sigma\kappa e^{-\kappa(t-s)} d\tilde{W}(s)\right] dt = \left[\kappa r^F(0,t) - \kappa r(t) + \int_0^t \sigma^2 e^{-\kappa(t-s)}[1 - e^{-\kappa(t-s)}]ds\right] dt \quad \textbf{(138)}$$

From this it follows that the process dr(t) from formula 2, taking into account the assumed volatility structures can be expressed as:

$$dr(t) = \left[r_t^F(0,t) + \kappa[r^F(0,t) - r(t)] + \int_0^t \sigma^2 e^{-2\kappa(t-s)} ds\right] dt + \sigma d\tilde{W}(t) \quad \textbf{(139)}$$

where this expression is found to be identical with formula 52 in the main text for $\kappa < 0$ and it is also identical with formula 14 in Bhar and Chiarella (1995 no. 53).

Here the well-known result that the Hull and White model is a Markovian model in the short rates is clearly shown by formula 7[32].

Let us now instead look at the following more general formulation for the forward rate volatility structure:

$$\sigma^F(t,T) = \sigma(t,T)G(r)(t)) \quad \textbf{(140)}$$

Assume now that $\sigma(t,T) = \sigma e^{-\kappa(T-t)}$ and $G(r(t)) = r(t)^v$ *for* for $v \geq 0$. We find that for $v = 0$ the form is identical to the volatility structure in the exponential decay model.

With these assumptions in mind, it is easily seen that the forward rates generally can be written as:

$$r^F(t,T) = r^F(0,T) + \int_0^t r(s)^{2v}\sigma^F(s,T)\sigma_P(s,T)ds + \int_0^t r(s)^v\sigma^F(s,T)d\tilde{W}(s) \quad \textbf{(141)}$$

If we now take the spot rate process as our starting point, we find:

$$r(t) = r^F(0,t) + \int_0^t r(s)^{2v}\sigma^F(s,t)\sigma_P(s,tT)ds + \int_0^t r(s)^v\sigma^F(s,tT)d\tilde{W}(s) \quad \textbf{(142)}$$

By applying the principles above, we can show that the stochastic process for the spot rate can be expressed as (here shown in abstract form):

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32 An alternative proof that precisely this volatility structure fulfills the Markov characteristic has been carried out in Caverhill (1994).

$$dr(t) = \left[r_t^F(0,t) + \partial_t \int_0^t G(r(s))^2 \sigma(s,t) \int_u^t \sigma(s,u) dy ds + \int_0^t G(r(t)) \partial_t \sigma(s,t) d\tilde{W}(s) \right] dt + G(r(t)) \sigma(t,t) d\tilde{W}(t) \quad \textbf{(143)}$$

On the basis of our assumptions on the functional forms of G(r(t)) and σ(t,T) respectively, we find that formula 11 can be written as:

$$dr(t) = \left[r_t^F(0,t) + \kappa[r^F(0,t) - r(t)] + \sigma^2 \int_0^t r(s)^{2v} e^{-2\kappa(t-s)} ds \right] dt + \sigma r(t)^v d\tilde{W}(t) \quad \textbf{(144)}$$

where this happens to be actually identical with formula 3.4a in Ritchken and Sankarasubramanian (1995).

Let us rewrite formula 12 as:

$$\begin{gathered} dr(t) = \left[r_t^F(0,t) + \kappa[r^F(0,t) - r(t)] + \sigma^2 \varphi(t) \right] dt + \sigma r(t)^v d\tilde{W}(t) \\ for \\ d\varphi(t) = [r(t)^{2v} - 2lakappa\varphi(t)]dt \end{gathered} \quad \textbf{(145)}$$

where this is the two-dimensional Markov model developed by Ritchken and Sankarasubramanian.

More general model formulations can of course also be made, for example by assuming that σ(t,T) is defined by $\sigma(t,T) = P_i(T - t)e^{-\kappa(T - ¨ t)}$ for $P_i(x) = \sum_{i=0}^{g_i - 1} \sigma_t x^i$ where g_i - 1 is the polynomial order. This is treated in more detail in Bhar and Chiarella (1995 no.53).

We have the following relationship between the price P(t,T) and the forward rates:

$$P(t,T) = \tilde{E}\left[\exp\left(-\int_t^T r^F(t,s) ds \right) \right] \quad \textbf{(146)}$$

In the following we will show that yield-curve at any time t is fully specified by the initial yield- curve, the spot rate and φ(t). From formula 9 we have the following relationship:

$$r^F(t,T) = r^F(0,T) + \int_0^t r(s)^{2v} \frac{\sigma^2}{\kappa} e^{-\kappa(T-s)}[1 - e^{-\kappa(T-s)}] ds + \sigma e^{-\kappa T} \int_0^t r(s)^v e^{\kappa s} d\tilde{W}(s) \quad \textbf{(147)}$$

From formula 10 we have:

$$\sigma e^{-\kappa t}\int_0^t r(s)^{\nu} e^{\kappa s} d\tilde{W}(s) = r(t) - r^F(0,t) - \int_0^t r(s)^{2\nu}\frac{\sigma^2}{\kappa}e^{-\kappa(t-s)}[1 - e^{-\kappa(t-s)}]ds \qquad \textbf{(148)}$$

such that formula 15 can be expressd as::

$$\begin{aligned} r^F(t,T) = r^F(0,T) &+ \int_0^t r(s)^{2\nu}\frac{\sigma^2}{\kappa}e^{-\kappa(T-s)}[1 - e^{-\kappa(T-s)}]ds + e^{-\kappa(T-t)}[r(t) - r^F(0,t)] \\ &- e^{-\kappa(T-t)}\int_0^t r(s)^{2\nu}\frac{\sigma^2}{\kappa}e^{-\kappa(t-s)}[1 - e^{-\kappa(t-s)}]ds \end{aligned} \qquad \textbf{(149)}$$

which can be simplified as follows:

$$r^F(t,T) = r^F(0,T) + e^{-\kappa(T-t)}[r(t) - r^F(0,t)] + \frac{\sigma^2}{\kappa}[e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}]\varphi(t) \qquad \textbf{(150)}$$

The price expression for P(t,T) now follows directly by inserting formula 18 into formula 14 and carrying out the integration. Formula 18 shows that the bond price (given σ and κ) is completely defined by the initial term structure and the two state variables r(t) og φ(t).