THE NORMAL CLASS OF ARBITRAGE-FREE SPOT-RATE MODELS - IMPLICATION AND IMPLEMENTATION

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Abstract: In this paper we first show how to determine the T-forward adjusted risk-measure using the concept of fundamental solution to linear PDE's. After that, relying on Fourier transformation we derive bond-and bond-option prices for the Extended Vasicek model from Hull and White (1990) and the Quadratic Interest Rate model. With respect to the Quadratic Interest Rate model we succeed in carrying the analysis much further than Jamshidian (1996). A special discrete time model - which in some cases is appropriate for the Quadratic Interest Rate model - is also derived.

The last part of the paper analyse Monte Carlo techniques in connection with spot-rate models with a time-dependent drift. We also introduce a method - using the concept of forward induction - that constrain the Monte Carlo simulated spot-rate process for the matching of the initial yield-curve. For the pricing of path-dependent contingent claim, we deduce that, even though Monte Carlo is the natural method to use, it might not be the most efficient one - at least not when the spot-rate is Markovian.

Keywords: Arbitrage-free pricing, spot-rate models, Fourier transformation, forward adjusted risk-measure, closed form solutions, lattice-models, SDE discretization schemes, Monte Carlo methods, Markovian structures, positive interest rates

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1: Introduction

When yield curve models are used in practice the traditional approach is to make sure that the parameters describing the process for the term structure satisfy certain requirements. The most important requirement in this connection is that the initial observed yield curve is exactly described by the chosen model - that is the model takes the initial yield curve as given. This has the desired characteristic that all other interest rate derivatives are priced by using a no-arbitrage argument.

The first yield curve model to satisfy this requirement was proposed by Ho and Lee (1986). Other models building on this basic idea are for example Black, Derman and Toy (1990), Black and Karasinski (1991) and Hull and White (1993a) - where these models are so called spot-rate models. The most general approach along these lines is the framework developed by Heath, Jarrow and Morton (1991), which build an m-factor continuous time term structure model in the instantanous forward rates.

It is of course preferable to price interest rate contingent claims analytically instead of having to use time-consuming numerical methods. Unfortunately, for many models this is not possible. In general, a normal distribution assumption is needed in order to have a rich analytical structure for the term structure model.

This assumption has that well known and (properly) undesired characteristic that interest rates can become negative. Recently, for that reason, a great deal of focus has been on models which contain some degree of tractability and at the same time do not allow for negative interest rates.

In the framework of HJM there is the log-normal Market Model from Brace, Gatarek and Musiela (1995) which is used by Miltersen, Sandmann and Sondermann (1995) to derive analytical expression for options on discount bonds. Another model in the HJM framework that precludes negative interest rates is the model from Flesaker and Houghton (1996) which builds a new equivalent martingale measure called the terminal measure in order to prevent negative interest rates.

In the class of spot-rate models there is the extended Cox, Ingersoll and Ross model from Jamshidian (1995) and the Quadratic interest rate model from Jamshidian (1996).

The first result of this paper is to - using the concepts of fundamental solutions - show an alternative/new method to determine the T-forward adjusted risk-measure for interest rate models which does not rely on the use of Girsanov's theorem. With respect to obtaining the fundamental solution of linear PDEs we will here introduce a new method relying on the

Fourier transformation.

Following along these lines we will focus on the one-factor Quadratic interest rate model, and in that connection we will show that the analytical structure of this model is very rich - nearly as rich as the extended Vasicek model from Hull and White (1993a). We will employ a different approach than the one advocated by Jamshidian which has that nice characteristic that we are capable of carrying the analysis of the Quadratic interest rate model much further[1]. We will in this paper also present a new derivation of the extended Vasicek model from Hull and White (1993a), which uses the concept of Fourier transformation.

The second result of this paper is the derivation of analytical expression for the prices of discount bonds, options on discount bonds - which, it appears, can be expressed in terms of the cumulative normal distribution. In the paper it will also be shown how the model can be fitted to the initial term structure in the spirit of Hull and White (1990).

For the pricing of American style securities (like for example Bermudan options) we will specify a special discrete time version for the model and compare it to the approach from Hull and White (1994).

The last part of the paper will focus on pricing techniques in order to handle path-dependent interest-rate derivatives. In connection with Monte Carlo simulation we will only focus on one-factor spot-rate processes with time-dependent coefficients. Monte Carlo simulation of yield-curve models that by construction match the initial term structures has (as far as we know) only been applied to models in the Heath, Jarrow and Morton framework (see for example Zhenyu (1994)). Recently, Andersen (1995) has designed a powerful application of importance sampling for simulating interest rates with time-homogenous parameters.

We will here design a procedure which ensures that a given Monte Carlo simulation method matches the initial yield-curve by construction - which is the third main result of the paper.

The paper is organized as follows: In section 2, we will define a general class of one-factor interest rate models and derive the partial differential equation that all interest rate contingent claims have to satisfy. In this section, we will also show how the T-forward adjusted risk-measure and fundamental solutions to PDEs are connected. In section 3, a method to obtain fundamental solutions from the PDE using the concept of Fourier transformation is introduced.

Section 4 will employ this technique for the extended Vasicek model and in that connection show how the price of discount bonds and options on discount bonds can be obtained. In section 5 we will turn our focus on the Quadratic interest rate model and show how to derive the price of discount bonds and options on discount bonds.

[1] This has independtly been done by Pelsser (1995).

We will in the spirit of Hull and White (1990) and Jamshidian (1996) show how it is possible to fit the model to the initial term structure. The next two sections will focus on pricing techniques when closed form solutions are not avaiable. First we will focus on a discrete time model for the Quadratic interest rate model and second we will analyse general Monte Carlo methods for spot-rate processes with time-dependent parameters.

2: Generalized one-factor models

In this paper, I consider a continuous trading economy with zero-coupon bonds and a money market account with a trading interval $[0,\tau]$, for a fixed $\tau > 0$. In addition, it is assumed that money does not exist, i.e. that the agents in the economy are forced at all times to invest all their funds in assets. As usual, the uncertainty in the economy is characterized by the probability space (Ω,F,P), where Ω is the entire state space, P is a probability measure and F is the event space. At the same time, it is assumed that a one-dimensional Wiener process exists: $W = [W(t);0 < t \leq T < \tau]$, with a drift equal to zero (0) and a variance equal to one (1).

Let us now consider, like Hull and White (1993a), the following rather general single-factor yield curve model:

$$\begin{aligned} dy &= \mu(\varphi(t),y,t)dt + \sigma(y,t)dW \\ &\textit{where} \\ r &= F(t,y) \end{aligned} \qquad \textbf{(1)}$$

Where $\mu(\bullet)$ represents the drift coefficient, which may be a function of time and the state variable, and where $\varphi(t)$ is a time-dependent function, $\sigma(\bullet)$ represents the diffusion coefficient, which may be a function of time and the state variable, and dW is a Wiener process with the following properties: $(dW)^2 = dt$, $dtdW = 0$ and $(dt)^2 = 0$. Furthermore, the spot rate is determined from the underlying process through the function F(t,y).

An important class of models arise under the assumption that the diffusion coefficient is independent of the state variable, i.e. $\sigma(y,t) = \sigma(t)$ and that the drift coefficient is defined as $\mu(\varphi(t),y,t) = \varphi(t) - \kappa(t)y$ - where $\kappa(t)$ is the mean-reversion parameter.

For this particular choice, the process y has a normal distribution. Therefore, I will denote this class of models as normal models. By making appropriate choices for F(t,y) it is possible to show that several existing models belong to this category of normal models. For $F(t,y) = y$ we obtain the extended Vasicek model from Hull and White (1990). By assuming $F(t,y) = e^y$ we get the Black and Karasinski (1991) model and for $F(t,y) = y^2$, this will lead to the one-factor Quadratic model from Jamshidian (1996). Lastly, (for $F(t,y) = y$) we get the Ho and Lee (1986) model for $\kappa(t) = 0$.

Using the arbitrage-free argument, it can be shown that the price of g(y,t,T) of an interest rate derivative security at time t which has a payoff at time T satisfies the partial differential equation:

$$g + g_y[\mu(\varphi(t),y,t) - \lambda\sigma(y,t)] + \frac{1}{2}\sigma(y,t)^2 g_{yy} - F(t,y)g = 0 \qquad \textbf{(2)}$$

Where λ denotes the market price of risk.

The price g(y,t,T) of an interest rate derivative that has a payoff H(y(T)) at maturity T can be calculated by solving the partial differential equation subject to the boundary condition g(y,T,T) = H(y(T)) at time T - that is at expiry/maturity the value of the derivative security is equal to the payoff at the time of expiry.

The Feynman-Kac solution to formula no. 2 (if it exists) can be written as[2]:

$$\begin{aligned} g(y,t,T) &= E^{Q}\left[\left(\exp-\int_t^T F(s,y(s))ds\right)H(y(T))|F_t\right] \\ &= E^{Q}\left[\left(\exp-\int_t^T r(s)ds\right)H(y(T))|F_t\right] \end{aligned} \qquad \textbf{(3)}$$

Where the expectation is taken, conditional on the information avaiable at time t, with respect to the risk-adjusted process:

$$\begin{aligned} dy &= \mu(\varphi(t),y,t) - a(t)y]dt + \sigma(y,t)d\tilde{W} \\ &\textit{where} \\ d\tilde{W} &= dW - \lambda\sigma(y,t) \end{aligned} \qquad \textbf{(4)}$$

Where $\tilde{W}_i$ is a Wiener process on (Ω,F,Q), for dQ = ρdP and ρ is the Radon-Nikodym derivative.

Using the Feynman-Kac formula the price of any interest rate derivative is given by the risk-adjusted expectation of the discounted payoff. However, as the discounting term and the payoff term are two correlated stochastic variables, this expectation is, in general, difficult to evaluate. As shown by Karoui, Myneni and Viswanathan (1993) it is more efficient to use the change of numerator techniques which in a sense moves the discounting outside the expectation and then the expectation is taken under a different probability measure and only with respect to the payoff at maturity. In connection which formula 3 this means that the price g(y,t,T) can be written as:

(5)

[2] In general the term " Feynman-Kac" is considered a misnomer as it originally refers to the probabilistic solution to a narrower class of parabolic equations than the Cauchy problem, see Duffie (1996) Appendix E. Typically formula 3 would be called the probabilistic solution to the PDE.

Where Q_T is the probability measure that is associated with using the T-period zero-coupon bond as numerator.

The result from formula 5 is obtained by using Girsanov's theorem. In the rest of this chapter, I will show an alternative derivation of this result, using the concept of fundamental solutions of PDEs.

Let us now suppose that a function d(y,t,T,z) exists that satisfies a linear PDE in y and t (that is a PDE like the one in formula 2), with boundary condition:

$$d(y,T,T,z) = \delta(y - z) \tag{6}$$

for all z and T. Formula 6 means that at time T the function d(y,t,T,z) will collapse into a Dirac delta-function[3] centered at point z. If we now consider a boundary condition described by the function H(y(T)) at time T then we have that:

$$g(y,t,T) = \int_{-\infty}^{\infty} H(z)d(y,t,T,z)dz \tag{7}$$

satisfies the partial differential equation, due to the linearity of the differentiation operator.

Furthermore, the above equation also satisfies the boundary condition at t = T. This is true since at t = T we have:

$$g(y,T,T) = \int_{-\infty}^{\infty} H(z)d(y,T,T,z)dz = \int_{-\infty}^{\infty} H(z)\delta(y - z)dz = H(y(T)) \tag{8}$$

where the last equality follows from the definition of the Dirac delta-function.

This means that for any given boundary condition, we can use the function d(y,t,T,z) to construct a solution to the PDE. For this reason, the function d(y,t,T,z) is called the fundamental solution to the partial differential equation.

It is actually possible to give an economic interpretation of formula 7. The Dirac delta-function can be thought of as the continuous time version of the payoff of an Arrow-Debreu security. As $\delta(y - z) \neq 0$ if only y = z and $\int_{-\infty}^{\infty} \delta(y - z)dz = 1$, we could say that the Dirac delta-function gives a payoff worth 1 in the state y = z and zero (0) elsewhere. This property means that d(y,t,T,z) can be viewed as the price at time t in state y of an Arrow-Debreu

[3] See Wilmott, Dewynne and Howison (1993) chapter 5.

security that has a payoff of 1 at time T in state y = z.

If we now consider a discount bond, it is well known that this is a security that has a payoff of 1 in all states at maturity T. From this follows that the price at time t of a zero-coupon bond that matures at time T (P(y,t,T))[4] is given by $P(y,t,T) = \int_{-\infty}^{\infty} d(y,t,T,z)dz$.

If the economy is arbitrage-free, the prices of Arrow-Debreu securities cannot become negative and prices of discount bonds are finite. These observations lead to another interpretation for formula 7. More precisely it is possible to write the following relationship:

$$p(y,t,T,z) = \frac{d(y,t,T,z)}{P(y,t,T)} \quad \textbf{(9)}$$

The function p(y,t,T,z) is positive and integrates by construction to 1 with respect to z. Any function that integrates to 1 and is non-negative can be interpretated as a probability density function. By this, it follows that formula 7 can be rewritten as:

$$g(y,t,T) = P(y,t,T)\int_{-\infty}^{\infty} H(z)p(y,t,T,z)dz = P(y,t,T)E^{Q_T}[H(z(T))|F_t] \quad \textbf{(10)}$$

Where the expectation is taken, conditional on the information avaiable at time t, with respect to the density p(y,t,T,z), the density p(y,t,T,z) will in general differ from the transition density of the risk-adjusted process of y from formula 4.

If we now combine this result with the relation from formula 3, where z represents y(T), we get the following result:

$$g(y,t,T) = E^{Q}\left[\left(\exp-\int_{t}^{T} r(s)ds\right)H(z)|F_t\right] = P(y,t,T)E^{Q_T}[H(z)|F_t]F_t] \quad \textbf{(11)}$$

That is, we have now expressed the Q-expectation as the discounted value of the Q_T-expectatation of the time T payoff. Under the expectation E^Q the prices $\frac{d(y,t,T,z)}{P(y,t,T)}$ of interest rate derivatives with maturity T denominated in T-period discount bonds are martingales. This also holds true under E^{Q_T} , since for each $t < T_1 < T$ we have:

[4] I will interchangely use the notation P(t,T), P(y,t,T) and P(q,t,T) for the price at time t for a disount bond that matures at time T - whenever applicable.

$$E^{Q_T}\left[\frac{g(y(T_1),T_1,T)}{P(y(T_1),T_1,T)}|F_t\right] = E^{Q_T}\left[\frac{P(y,T_1,T)E^{Q_T}[H(y(T))|F_{T_1}]}{P(y,T_1,T)}|F_t\right] = E^{Q_T}[E^{Q_T}[H(y(T))|F_{T_1}]|F_t]$$
$$= E^{Q_T}[H(y(T))|F_t] = \frac{g(y(t),t,T)}{P(y(t),t,T)} \quad \textbf{(12)}$$

Where the fourth equality follows from the law of iterated expectations, and the second and fifth are obtained through the use of formula 10.

3: Deriving Fundamental Solutions to linear PDEs

When we want to find the fundamental solution to a linear PDE, we are actually seeking functions d(y,t,T,z) that satisfy the partial differential equation and at expiry/maturity collapse into the Dirac delta-function for all z. Instead of trying to obtain the fundamental solution directly, I will solve the partial differential equation for the Fourier transformation of the fundamental solution and find the fundamental solution from here.

Let us now consider the function D(y,t,T,ς) defined as:

$$D(y,t,T,\varsigma) = \int_{-\infty}^{\infty} e^{-i\varsigma z} d(y,t,T,z)dz \quad \textbf{(13)}$$

Where this formula represents the Fourier transformation in the variable z and d(y,t,T,z), where i is the imaginary number, for which $i^2 = -1$. The function D(y,t,T,ς) satisfies the same PDE as d(y,t,T,z) but the boundary condition is here given by:

$$D(y,T,T,\varsigma) = \int_{-\infty}^{\infty} e^{-i\varsigma z}\delta(y - z)dz = e^{-i\varsigma y} \quad \textbf{(14)}$$

Where the last equality follows from the definition of the Dirac delta-function. It follows from here that the boundary condition for the function D(y,t,T,ς) is of a very simple form.

We can obtain the fundamental solutions d(y,t,T,z) by inverting the Fourier transformation D(y,t,T,ς). This inversion is usually very simple if we use the following property. Using the relation from formula 9, we can write the fundamental solution as a product of the price of the T-period discound bond and a probability density function, hence we can rewrite formula 14 as:

$$D(y,tT,T,\varsigma) = \int_{-\infty}^{\infty} e^{-i\varsigma z}P(y,t,T)p(y,t,T,z)dz = P(y,t,T)\tilde{p}(y,t,T,\varsigma) \quad \textbf{(15)}$$

Where $\tilde{p}(y,t,T,\varsigma)$ is the Fourier transformation of the density function p(y,t,T,z). Apart from a minus sign in the exponent (and, possibly, a factor $\sqrt{\frac{1}{2\pi}}$), characteristic functions coincide with Fourier transforms of a probability density function in the continuous case and with Fourier series in the discrete case. Furthermore, it is well known that it is possible to obtain the moments of the underlying distribution directly from the charateristic function[5].

If we assume that $\varsigma = 0$, then we have that the boundary condition for $D(y,t,T,\varsigma)$ becomes identical to the boundary condition for a discount bond, hence:

$$D(y,t,T,0) = P(y,t,T) \Rightarrow \tilde{p}(y,t,T,\varsigma) = \frac{D(y,t,T,\varsigma)}{D(y,t,T,0)} \qquad \textbf{(16)}$$

Where the last equality follows directly from formula 14.

From formula 16 it can be seen that from $D(y,t,T,\varsigma)$ we can obtain the bond price P(y,t,T) and the Fourier transformation of the density function p(y,t,T,z) - that is $\tilde{p}(y,t,T,\varsigma)$ -and then we can derive the density p(y,t,T,z) by inversion of the characteristic function[6]. Once we have defined the bond price P(y,t,T) and the transitions densities p(y,t,T,z), it is possible to price interest rate derivative securities by using formula 10.

Before turning my attention to the Quadratic interest rate model, I will demonstrate this technique for a more simple model, namely the extended Vasicek model from Hull and White.

4: Deriving the Fundamental Solution for the Extended Vasicek Model

As was mentioned in section 2, we get the extended Vaseicek model from Hull and White (1990) by making the following specification of the drift-parameter, the diffusion-parameter and the function F(t,y):

$$\begin{aligned} F(t,y) &= y \\ \mu(\varphi(t),y,t) &= \varphi(t) - \kappa(t)y \\ \sigma(y,t) &= \sigma(t) \end{aligned} \qquad \textbf{(17)}$$

which results in the following risk-adjusted spot rate process:

[5] See Gut (1995).

[6] If it is not possibly to invert the characteristic function then we can at least get useful information about the distribution, such as the moments or approximating distributions.

$$dy = [\varphi(t) - \kappa(t)y]dt + \sigma(t)d\tilde{W} \quad \textit{where} \quad d\tilde{W} = dW - \lambda\sigma(t) \tag{18}$$

Let me employ the following change of variable $r = y + \psi(t)$ $\varphi(t) = \psi'(t) + \kappa(t)\psi(t)$ and $r(0) = \psi(0)$ as $y(0) = 0$. Using Ito-lemma we can deduce that

$$\psi(t) = e^{-k(t)}\int_0^t \varphi(s)e^{k(s)}ds \textit{ for } k(t) = \int_0^t \kappa(s)ds \quad .$$

Under this change of variable the partial differential equation from formula 2 becomes of the following form:

$$g_t - g_y[\kappa(t)y + \lambda\sigma(t)] + \frac{1}{2}\sigma(t)^2 g_{yy} - g[y + \psi(t)] = 0 \tag{19}$$

In order to obtain the price of a discount bond in this model, we have to solve the PDE in formula 19 with respect to the boundary condition $D(y,T,T,\varsigma) = e^{i\varsigma y}$.

We guess that the solution is of the affine exponential type[7]:

$$D(y,t,T,\varsigma) = e^{B(t,T,\varsigma) + A(t,T,\varsigma)y} \tag{20}$$

By substituting this functional form into the PDE we get[8]:

$$B_t + A_t y - A\lambda\sigma(t)] - A\kappa(t)y + \frac{1}{2}\sigma^2(t)A^2 - y - \psi(t) = 0 \tag{21}$$

The partial differential equation can be solved by simultaneously solving the following two ordinary differential equations with respect to each other subject to the boundary conditions $A(T,T,\varsigma) = i\varsigma$ and $B(T,T,\varsigma) = 0$:

$$B_t \mp Ax(t) + \frac{1}{2}\sigma^2(t)A^2 - \psi(t) = 0 \qquad A_t - A\kappa(t) - 1 = 0 \quad \textit{for} \quad x(t) = \kappa(t) - \lambda\sigma(t) \tag{22}$$

[7] "Guess" is probably the wrong term to use in this context, as we know from Hull and White (1990) that this actually is the case.

[8] I will in this paper interchangely use the notation $A(t,T,\varsigma)$ and A, $B(t,T,\varsigma)$ and B and $C(t,T,\varsigma)$ and C - whenever applicable.

The solution for A(t,T,ς) and B(t,T,ς) can be shown to be given by[9]:

$$A(t,T,\varsigma) = -A(t,T) + i\varsigma D(t,T)$$

$$B(t,T,\varsigma) = B(t,T) - \int_t^T \left[i\varsigma D(s,T)[x(s) + \sigma^2(s)A(s,T)] + \frac{1}{2}\varsigma^2 D(s,T)^2\sigma^2(s) \right] ds$$

where

$$D(t,T) = e^{-\int_t^T a(s)ds}$$

$$A(t,T) = \int_t^T D(t,s)ds \qquad \textbf{(23)}$$

$$B(t,T) = \int_t^T \left[\frac{1}{2}\sigma^2(s)A(s,T)^2 \pm x(s)A(s,T) - \psi(s) \right] ds$$

Substituting this into formula 22 yields:

$$D(y,t,T,\varsigma) = e^{B(t,T) - A(t,T)y + i\varsigma M(t,T) - \frac{1}{2}\varsigma^2 V(t,T)}$$

where

$$M(t,T) = D(t,T)y - \int_t^T D(s,T)[x(s) \pm \sigma^2(s)A(s,T)]ds \qquad \textbf{(24)}$$

$$V(t,T) = \int_t^T D(s,T)^2\sigma^2(s)ds$$

From the boundary condition for D(y,t,T,ς) we get the boundary condition for discount bonds for ς = 0 and then we immediately obtain the price of a discount bond from equation 24, namely:

$$P(t,T) = e^{B(t,T) - A(t,T)y} \qquad \textbf{(25)}$$

and furthermore we see that the characteristic function of the density p(y,t,T,z) is given by:

$$\tilde{p}(y,t,T,\varsigma) = e^{i\varsigma M(t,T) - \frac{1}{2}\varsigma^2 V(t,T)} \qquad \textbf{(26)}$$

It can now be observed that formula 26 actually is the characteristic function for the normal distribution with mean M(t,T) and variance V(t,T). We therefore have that the fundamental

[9] See Appendix A where the actual derivation is done.

solution d(y,t,T,z) for the extended Vasicek model is given by:

$$d(y,t,T,z) = P(y,t,T)\frac{1}{\sqrt{2\pi V(t,T)}}e^{-\frac{1}{2}\frac{[z - M(t,T)]^2}{V(t,T)}} \quad \textbf{(27)}$$

Now that we have the fundamental solution, it is possible to price interest rate derivatives with the joint use of formula 10, formula 16 and the result from formula 27.

To illustrate this, let me show how to derive the price at time t of a call-option with expiry date T^F, for $t < T^F < T$, written on a discount bond which matures at time T. By assuming that $y(T^F) = z$ we can write the payoff as:

$$H(z) = [e^{B(T^F,T) - A(T^F,T)z} - K]^+ \quad \textbf{(28)}$$

Where K is the strike-price.

The price of the option can now be written as:

$$C(t,T^F) = P(t,T^F)\int_{-\infty}^{\infty}\left[e^{B(T^F,T) - A(T^F,T)z} - K\right]^+ \frac{1}{\sqrt{2\pi V(t,T^F)}}e^{-\frac{1}{2}\frac{[z - M(t,T^F)]^2}{V(t,T^F)}}\right]dz \quad \textbf{(29)}$$

From formula 28, it can be deduced that the payoff H(z) is positive for all z that satisfies $z < \frac{B(T^F,T) - \ln K}{A(T^F,T)}$.

If the integration in formula 29 is done over the region for positive values of z, we can write the price of the call-option in terms of the cumulative normal distribution, hence:

$$C(t,T^F) = P(t,T^F)e^{B(T^F,T) - A(T^F,T)M(t,T^F) + \frac{1}{2}A(T^F,T)^2V(t,T^F)}N\left(\frac{\frac{B(T^F,T) - \ln K}{A(T^F,T)} - M(t,T^F) + A(T^F,T)V(t,T^F)}{\sqrt{V(t,T^F)}}\right) - P(t,T^F)KN\left(\frac{\frac{B(T^T,T) - \ln K}{A(T^F,T)} - M(t,T^F)}{\sqrt{V(t,T^F)}}\right) \quad \textbf{(30)}$$

As $P(T^F,T) = e^{B(T^F,T) - A(T^F,T)M(t,T^F) + \frac{1}{2}A(T^F,T)^2V(t,T^F)}$ we get[10]

[10] Later in the paper, this will become obvious, see section 6 and Appendex E.

$$C(t,T^F) = P(t,T)N(d_1) - P(t,T^F)KNt(d_1 - A(T^F,T)\sqrt{V(t,T^F)})$$
$$\textit{where}$$
$$d_1 = \frac{\ln\left(\frac{P(t,T)}{P(t,T^F)K}\right)}{A(T^F,T)\sqrt{V(t,T^F)}} + \frac{1}{2}A(T^F,T)\sqrt{V(t,T^F)} \qquad \textbf{(31)}$$

where the expression in formula 31 also can be found in Hull and White (1993c). The price of a put-option is now easily derived by using either the put-call parity or the symmetric properties of the normal distribution.

Now I will turn my attention to the Quadratic interest rate model, and in the next section show how the price of a discount bond can be derived using the principle of fundamental solutions. Furthermore, I will show how to derive the price of a call-option on a discount bond in a easier expression than the one given in Jamshidian (1996) as it turns out that it can be expressed in terms of the cumulative normal distribution. I will then briefly discuss the pricing of caps, swaptions and options on coupon bonds in the Quadratic interest rate model compared to the extended Vasicek model.

After that I will discuss a discrete time implementation of the Quadratic interest rate model in comparison with the Hull and White (1994) trinomial approach. In section 8 I will focus on Monte Carlo simulation of spot-rate processes with time-dependent parameters.

5: The Quadratic Interest Rate Model

As mentioned in section 2, we get the Quadratic interest rate model by assuming that F(t,y) = y^2. Furthermore, I will only allow for one time-dependent parameter in the SDE for y, that is:

$$dy = [\varphi(t) - \kappa y]dt + \sigma d\tilde{W}$$
$$\textit{where} \qquad \textbf{(32)}$$
$$d\tilde{W} = dW - \lambda\sigma$$

This property to allow only for one time-dependent parameter in the stochastic differential equation and use that for matching the initial term structure is also the approach advocated by Hull and White (1994)[11].

Using the arbitrage-free argument, it can be shown that the price of g(y,t,T) of an interest rate

[11] In the case of the Quadratic Interest rate model we actually need a deterministic volatility for the underlying state-variable - this issue is addressed in Appendix F.

derivative security at time t which has a payoff at time T satisfies the partial differential equation:

$$g + g_y[\varphi(t) - \kappa y - \lambda\sigma] + \frac{1}{2}\sigma^2 g_{yy} - F(t,y)g = 0 \qquad \textbf{(33)}$$

We will employ the following change of variable[12]:

$$\begin{aligned} q &= y - \psi(t) \\ \psi(t) &= e^{-\kappa t}\left(\sqrt{r} + \int_0^t e^{\kappa s}\varphi(s)ds\right) \end{aligned} \qquad \textbf{(34)}$$

Under the change of variable, we have that the spot rate is governed by the following SDE:

$$\begin{aligned} dq &= -\kappa q dt + \sigma d\tilde{W} \\ &\textit{where} \\ r &= [q + \psi(t)]^2 \\ d\tilde{W} &= dW - \lambda\sigma \end{aligned} \qquad \textbf{(35)}$$

Which means that the price of g(y,t,T) of an interest rate derivative security at time t has to satisfy the following PDE:

$$g_t - g_q[\kappa q + \lambda\sigma] + \frac{1}{2}\sigma^2 g_{qq} - g[q + \psi(t)]^2 = 0 \qquad \textbf{(36)}$$

From the last sections, we know that in order to get a solution for an interest rate derivative security, we "just" have to find the expression for D(q,t,T,ς) as the price of discount bonds and the density p(q,t,T,z) can be found by inverting the characteristic function obtained through a Fourier transformation.

Using the technique employed in the last section we postulate that the functional form for D(q,t,T,ς) is given by:

$$D(q,t,T,\varsigma) = e^{A(t,T,\varsigma) - B(t,T,\varsigma)q - C(t,T,\varsigma)q^2} \qquad \textbf{(37)}$$

which satisfies formula 36 subject to the boundary condition $D(q,T,T,\varsigma) = e^{i\varsigma q}$.

Plucking formula 37 into formula 36 we have that the partial differential equation can be

[12] As will be apparent later in this paper, this change of variable will make a discrete time version of the model easier to implement.

solved by simultaneously solving the following three ordinary differential equations with respect to each other subject to the boundary conditions A(T,T,ς) = C(T,T,ς) = 0 and B(T,T,ς) = -iς:

$$\begin{aligned} A_t + \frac{1}{2}\sigma^2 B^2 - \sigma^2 C - \psi(t)^2 &= 0 \\ B_t - \kappa B - 2\sigma^2 BC + 2\psi(t) &= 0 \\ C_t - 2\kappa C - 2\sigma^2 C^2 + 1 &= 0 \end{aligned} \tag{38}$$

The solution for A(t,T,ς), B(t,T,ς) and C(t,T,ς) can be shown to be given by[13]:

$$\begin{aligned} C(t,T,\varsigma) &= C(t,T) \\ B(t,T,\varsigma) &= B(t,T) - i\varsigma D(t,T) \\ A(t,T,\varsigma) &= A(t,T) - \frac{1}{2}\sigma^2 \int_t^T [2i\varsigma D(s,T)B(s,T) + \varsigma^2 D(s,T)^2]ds \end{aligned} \tag{39}$$

Where:

$$\begin{aligned} A(t,T) &= \int_t^T \left[\frac{1}{2}\sigma^2 B(s,T)^2 - \sigma^2 C(s,T) - \psi(s)^2\right]ds \\ B(t,T) &= 2\int_t^T \frac{e^{\gamma s}h(s,T)}{e^{\gamma t}h(t,T)}\psi(s)ds \\ C(t,T) &= \frac{e^{2\gamma(T-t)} - 1}{h(t,T)} \\ D(t,T) &= \frac{2\gamma e^{\gamma(T-t)}}{h(t,T)} \\ &\textit{where} \\ \gamma &= \sqrt{\kappa^2 + 2\sigma^2} \\ h(t,T) &= (\kappa + \gamma)e^{2\gamma(T-t)} + (\gamma - \kappa) \end{aligned} \tag{40}$$

Substituting this into formula 37 yields:

[13] See Appendix B for the proof.

$$D(q,t,T,\varsigma) = e^{A(t,T) - B(t,T)q - C(t,T)q^2 + i\varsigma M(t,T) - \frac{1}{2}\varsigma^2 V(t,T)}$$

where

$$M(t,T) = D(t,T)q - \sigma^2 \int_t^T D(s,T)B(s,T)ds \qquad \textbf{(41)}$$

$$V(t,T) = \sigma^2 \int_t^T D(s,T)^2 ds = \sigma^2 C(t,T)$$

From the boundary condition of D(q,t,T,ς), we can derive the price for a zero-coupon bond, and we obtain:

$$P(t,T) = e^{A(t,T) - B(t,T)q - C(t,T)q^2} \qquad \textbf{(42)}$$

As in the extended Vasicek case, we see from equation 41 that the terms containing ς can be recognized as the characteristic function for the normal distribution. This leads us to conclude that under the T-forward adjusted probability measure the probability density function p(q,t,T,z) is equal to a normal probability function, which means that the fundamental solution d(q,t,T,ς) for the Quadratic interest rate model can be expressed as:

$$d(q,t,T,z) = P(q,t,T)\frac{1}{\sqrt{2\pi V(t,T)}} e^{-\frac{1}{2}\frac{[z - M(t,T)]^2}{V(t,T)}} \qquad \textbf{(43)}$$

That is the density for the Quadratic interest rate model is identical to the density for the extended Vasicek model, except for the definition of M(t,T) and V(t,T) - as they are both of the normal class.

As in section 4, we are now in a position were we can price interest rate derivatives by a joint use of the density from formula 43 and formulas 10 and 16.

To illustrate this, let me again focus on the pricing at time t of a call-option with expiry date T^F, for $t < T^F < T$, written on a discount bond which matures at time T. By assuming that $y(T^F) = z$ we can write the payoff as:

$$H(z) = [e^{A(T^F,T) - B(T^F,T)z - C(T^F,T)z^2} - K]^+ \qquad \textbf{(44)}$$

Where K is the strike-price.

The price of the option can now be written as:

$$C(t,T^F) = P(t,T^F)\int_{-\infty}^{\infty}\left[\left[e^{A(T^F,T) - B(T^F,T)z - C(T^F,T)z^2} - K\right]^+ \frac{1}{\sqrt{2\pi V(t,T^F)}} e^{-\frac{1}{2}\frac{[z - M(t,T^F)]^2}{V(t,T^F)}}\right]dz \qquad \textbf{(45)}$$

If we integrate formula 45 for all z for which the payoff H(z) is positive, we will be able to express the price of the call-option in terms of the cumulative normal distribution.

It follows directly from formula 44 that z is defined as the root in a second order equation, more precisely we find that z is positive for values of z lying in the following interval:

(46)

From this relation it it clearly seen that for Dis ≤ 0, the payoff H(z) will never be positive - that is the value of the option has to be equal to zero (0). If instead Dis > 0, we can integrate formula 45 over the region Lower < z < Upper. If this is done, we obtain the following analytical expression for the price of a call-option[14]:

$$\begin{aligned} C(t,T^F) &= P(t,T)\left[N\left(\frac{UpperF - E}{\sqrt{V(t,T^F)F}}\right) - N\left(\frac{LowerF - E}{\sqrt{V(t,T^F)F}}\right)\right] \\ &- P(t,T^F)K\left[N\left(\frac{Upper - M(t,T^F)}{\sqrt{V(t,T^F)}}\right) - N\left(\frac{Lower - M(t,T^F)}{\sqrt{V(t,T^F)}}\right)\right] \end{aligned} \qquad \textbf{(47)}$$

$$\textit{where}$$

$$E = M(t,T^F) - B(T^F,T)V(t,T^F)$$

$$F = 1 + 2C(T^F,T)V(t,T^F)$$

The same procedure can be used to derive the formula for a put-option, see Appendix C.

Quadratic interest rate models have also been analysed in for example Leblon and Moraux (2009) and Leippold and Wu (2001). However, they only consider the quadratic interest rate model for the case of constant parameters - that is their representations is not defined so it by construction match the initial yield-curve as is the case here.

Earlier examples of quadratic models - for the case of constant parameters - is the double square-root model of Longstaff (1989), the extended (and corrected) version from Beaglehole and Tenney (1991) and Beaglehole and Tenney (1992). One can also mention the SAINTS model of Constantinides (1992), where the pricing kernel is being defined as exogenously specified as a time-seperable quadratic function of the Markov process.

The paper (that to our knowledge) takes the analyses of the quadratic interest rate model

[14] See Appendix C where the derivation has been made.

furtherst is Leippold and Wu (2001). They for example manage to obtain closed-form solution for European options on bonds, which however require that one has to perform a numerical inversion of a Fourier transform. As comparison Jamshidian's (1996) expression for the price of an option on a discount bond was expressed in terms of the non-central chi-square distribution. See also Andersen and Piterbarg (2010), section 12.3.

As it is possible to get a (nearly) closed form solution for both the price of a discount bond and the price of an option on a discount bond in the Quadratic interest rate model, it follows directly that a whole range of instruments can also be priced analytically. As examples, I can mention coupon bonds, Calls/Floors and Swaps. This is of course also possible in the extended Vasicek model but in the extended Vasicek model we are also capable of deriving closed form solution for the value of swaptions and options on coupon bonds, see Jamshidian (1989). In order to value swaptions or options on coupon bonds in the Quadratic interest model it does not seem possible to derive closed form solutions, instead we could use numerical integration in order to solve the following equation:

$$C(t,T^F) = P(t,T^F)\int_{-\infty}^{\infty}[P^k(T^F) - K]^+ p(y,t,T,z)dz$$
$$\text{where} \qquad \textbf{(48)}$$
$$P^k(t) = \sum_{j=1}^{n} F_j P(t,T_j)$$

where formula 48 gives the price at time t of a call-option that expires at time T^F, written on a coupon bond that matures at time T_n, for $t < T^F < T_j$ and $j \in \{1,2,....,n\}$.

Another possibility is to discretize the process in the manner of Hull and White (1994) and then price interest rate contingent claims by backwardation. This procedure is quite flexible and allows us to price claims with American style features[15] (backward path-dependency), so for that reason I will show how to employ this technique for the Quadratic interest rate model - this is done in section 7.

6: Arbitrage-free pricing in the Quadratic interest rate model

The reason for incorporating a time-dependent parameter in the drifts-specification as was done for the extended Vasicek model in formula 18 and for the Quadratic interest rate model in formula 32, was to be able to use the initial term structures as input and then price all other instruments using a no-arbitrage argument. The purpose of this section is to show how $\psi(t)$ is to be chosen so that the model by construction matches the initial term structure.

I will here only concentrate on the Quadratic interest model as the extended Vasicek model

[15] For example Bermudan options.

with respect to this reformulation has been analysed in Kijima and Nagayama (1995).

The Heath, Jarrow and Morton framework is my starting point here as this framework by construction matches the initial term structure, and then I will rewrite the Quadratic interest rate model so it is consistent with the procedure of HJM.

Let me as HJM start by defining the forward rates:

$$r^F(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} \quad for\ all\ t < T \tag{49}$$

We have that P(t,T) can be expressed as:

$$P(t,T) = E^Q\left[\left(\exp-\int_t^T r(s)ds\right)|F_t\right] \tag{50}$$

Plucking this expression into formula 49 yields:

$$\begin{aligned} r^{FP}(t,T) &= \frac{\partial E^Q\left[\left(\exp-\int_t^T r(s)ds\right)|F_t\right]}{\partial T}\frac{1}{P(t,T)} = E^Q\left[\frac{\partial \exp-\int_t^T r(s)ds}{\partial T}|F_t\right]\frac{1}{P(t,T)} \\ &= E^Q\left[\left(\exp-\int_t^T r(s)ds\right)r(T)|F_t\right]\frac{1}{P(t,T)} = P(t,T)E^{Q_T}[r(T)|F_t] \end{aligned} \tag{51}$$

where the last part in formula 51 follows directly from equation 5. As a consequence, it is obvious that under the Q_T- probability measure the expected instantaneous forward rate at time T is given by the spot rate at time T.

We have that $r(T) = [q(T) + \psi(T)]^2$ and furthermore, we know from the last equation under the T-adjusted probability measure the instantanous forward rate is equal to the Q_T-expected spot rate, hence we can write $r^F(t,T)$ as[16]:

$$r^F(t,T) = \left[\psi(T) + D(t,T)q - \sigma^2\int_t^T B(s,T)D(s,T)ds - \frac{1}{2}V(t,T)\right]^2 + \frac{1}{2}V(t,T) \tag{52}$$

[16] In Appendix E the approach that leads to this result is shown with respect to the extended Vasicek model.

For t = 0 we get that q(0) = 0 because $\psi(0)^2 = r(0)$, which means:

$$r^F(0,T) = \left[\psi(T) - \sigma^2\int_0^T B(s,T)D(s,T)ds - \frac{1}{2}V(0,T)\right]^2 + \frac{1}{2}V(0,T) \qquad \textbf{(53)}$$

By re-arranging terms in formula 53 we find that $\psi(T)$ is of the following form:

$$\begin{aligned}\psi(T) &= F(T) + \sigma^2\int_0^T D(s,T)B(s,T)ds \\ F(T) &= \sqrt{r^F(0,T) - \frac{1}{2}V(0,T)} + \frac{1}{2}V(0,T)\end{aligned} \qquad \textbf{(54)}$$

Which is only valid when $r^F(0,T) \geq \frac{1}{2}V(0,T)$.

This property clearly means that the model cannot be fitted to all initial term structures if $\frac{1}{2}V(0,T) > r^F(0,T)$.

From the last section, we know that B(t,T) is a function of $\psi(T)$, hence formula 54 is an integral equation in $\psi(T)$. In Appendix D, it is in this respect shown that it is possible to express $\psi(T)$ as:

$$\psi(T) = F(T) + 2e^{-\kappa T}\int_0^T e^{\kappa s)}V(0,s)F(s)ds \qquad \textbf{(55)}$$

Now, following the approach in Jamshidian (1996), it is more efficient to express A(t,T) and B(t,T) in formula 40 in terms of the initial term structure and B(0,T) and A(0,T). As B(t,T) and A(t,T) are specified for now, we are in a position where we have to use a numerical integrations technique for every value we need of A(t,T) and B(t,T).

In the rest of this section, I will therefore show how to evaluate $P(T_1,T)$ in terms of $P(0,T_1)$ and P(0,T).

Proposition no. 1

We find generally that $A(T_1,T)$, $B(T_1,T)$ and $C(T_1,T)$ at time t for $0 < t < T_1 < T$ can be written as:

$$A(T_1,T) = [A(t,T) - A(t,T_1)] - \frac{1}{2}\ln F - \left[\frac{\frac{1}{2}B(T_1,T)^2V(t,T_1) + B(T_1,T)G - C(T_1,T)G^2}{F}\right]$$

$$B(T_1,T) = \frac{[B(t,T) - B(t,T_1)]F}{D(t,T_1)} + 2C(T_1,T)G$$

$$C(T_1,T) = \frac{C(t,T) - C(t,T_1)}{D(t,T_1)^2 - 2V(t,T_1)[C(t,T) - C(t,T_1)]} \qquad \textbf{(56)}$$

where

$$F = 1 + 2C(T_1,T)V(t,T_1)$$

$$G = \sigma^2\int_t^{T_1} D(s,T_1)B(s,T_1)ds$$

For t = 0 we find that A(0,T), B(0,T) and C(0,T):

$$A(0,T) = \ln P(0,T)$$

$$B(0,T) = 2\int_0^T D(0,s)\sqrt{r^F(0,T) - \frac{1}{2}V(0,s)}ds \qquad \textbf{(57)}$$

$$C(0,T) = C(0,T)$$

Proof:

We have the following property (see Appendix E):

$$P(t,T) = P(t,T_1)E^{Q_{T_1}}[P(T_1,T)|F_t] \qquad \textbf{(58)}$$

Plucking in the expression for P(t,T) from formula 37 and the definition for $E^{Q_{T_1}}[P(T_1,T)|F_t]$ *from formula 5 in Appendix C given* $T^F = T_1$ *and re-arranging terms we get:*

$$e^{A(t,T) - A(t,T_1) - [B(t,T) - B(t,T_1)]q - [C(t,T) - C(t,T_1)]q^2}$$

$$= \sqrt{F}e^{A(T^F,T) + \left[\frac{\frac{1}{2}B(T_1,T)^2V(t,T_1) - B(T_1,T)M(t,T_1) - C(T_1,T)M(t,T_1)^2}{F}\right]} \qquad \textbf{(59)}$$

If we use the definition of $M(t,T_1)$ *from equation 41 (given* $T^F = T_1$*) and collecting terms of equal power in q, we get:*

$$A(t,T) - A(t,T_1) = A(T_1,T) + \frac{1}{2}\ln F + \left[\frac{\frac{1}{2}B(T_1,T)^2V(t,T_1) + B(T_1,T)G - C(T_1,T)G^2}{F}\right]$$
$$B(t,T) - B(t,T_1) = \frac{D(t,T_1)[B(T_1,T) - 2C(T_1,T)G]}{F} \quad \textbf{(60)}$$
$$C(t,T) - C(t,T_1) = \frac{C(T_1,T)D(t,T_1)^2}{F}$$

Now equation 56 follows directly from here.

For t = 0 we have for all T > 0 that A(0,T) = ln P(0,T) (as q = 0 for t = 0) and we also have that $G_0 = \psi(T) - F(T)$, using this we can rewrite formula 60 as:

$$A(T_1,T) = \ln\left(\frac{P(0,T)}{P(0,T_1)}\right) - \frac{1}{2}\ln F_0$$
$$- \left[\frac{\frac{1}{2}B(T_1,T)^2V(0,T_1) + B(T_1,T)[\psi(T_1) - F(T_1)] - C(T_1,T)[\psi(T_1) - F(T_1)]^2}{F_0}\right] \quad \textbf{(61)}$$
$$B(T_1,T) = \frac{[B(0,T) - B(0,T_1)]F_0}{D(0,T_1)} + 2C(T_1,T)[\psi(T_1) - F(T_1)]$$
$$C(T_1,T) = \frac{C(0,T) - C(0,T_1)}{D(0,T_1)^2 - 2V(0,T_1)[C(0,T) - C(0,T_1)]}$$

where F_0 is F evaluated at t = 0 and G_0 is G evaluated at t = 0.

The only unknown factor in this relationsship is B(0,T) for all T. In order to solve this problem we will employ the same technique we used in Appendix E with respect to the extended Vasicek model.

From equation 53 we have an expression for $r^F(0,T)$, but we are also able to describe the instantanous forward rate by using formula 49, hence:

$$-A_T(0,T) + B_T(0,T)q + C_T(0,T)q^2 = \left[\psi(T) + D(0,T)q - G_0 - \frac{1}{2}V(0,T)\right]^2 + \frac{1}{2}V(0,T) \quad \textbf{(62)}$$

Using formula 62 and collecting terms of equal power in q we find that $B_T(0,T)$ can be written as follows:

$$B_T(0,T) = 2\psi(T)D(0,T) - 2D(0,T)G_0 - D(0,T)V(0,T() \quad \textbf{(63)}$$

Using the knowledge that $G_0 = \psi(T) - F(T)$, we can simplify the expression for $B_T(0,T)$ as:

$$B_T(0,T) = 2D(0,T)\sqrt{r^F(0,T) - \frac{1}{2}V(0,T)} \qquad \textbf{(64)}$$

Formula 57 now follows, which completes the argument.

Qed.

It follows from this that it is only necessary to valuate B(0,T) for all T - that is we can store the calculations.

7: A Discrete Time Version of the Quadratic Interest Rate Model

If we want to price instruments with American style features - that is path-dependent securities, or we need to price European style instruments where no analytical expression is available we can build a discrete model.

One possibility is to discretize the PDE using the trinomial model from Hull and White (1994) which of couse is directly applicable for the Quadratic model. It should nevertheless with respect to the Hull and White trinomial model be mentioned that it might not be possible to use the analytical expression for the arbitrage-free bond prices in connection with the trinomial approach[17].

What I mean is that if for example you want to price a 3-year option on a 10-year bond you might have to build the lattice all the way up to 10-years instead of just to 3-years. The reason for that is that the Hull and White trinomial model is modelled in the Δt-interest rate whereas in the analytical bond price expression you use the continous time interest rate. This difference in perspective means that you have to convert the Δt-interest rate given in the lattice to the continous time interest rate that you need to use in the analytical expression for the bond price. This conversion is in many cases not possible, as the continous time interest rate is the solution to a second order equation which might not have a positive discriminant.

For that reason, we will define a new method for building discrete time models for one-factor interest rate models. The idea is to build a lattice in the continous time interest rate instead of the Δt-interest rate.

We will construct a grid which is defined in changes in q and t. The steps along the y-axis will be Δq and along the x-axis Δt. We will define a node (i,j) on the lattice as a point where $t_i = i\Delta t$ and $q_j = j\Delta q$. Every node on the grid will be an approximation to the true value of the

[17] Which in general is a very efficient method.

security, and will for node (i,j) be denoted as $g_{i,j}(t,q)$. From Clewlow (1992) we have that in the explicit finite difference technique the partial derivatives of g(t,q) at node (i,j) are approximated as follows:

$$\begin{aligned} g_q(t,q) &\approx \frac{g_{i+1,j+1}(t,q) - g_{i+1,j-1}(t,q)}{2\Delta q} \\ g_t(t,q) &\approx \frac{g_{i+1,j}(t,q) - g_{i,j}(t,q)}{\Delta t} \\ g_{qq}(t,q) &\approx \frac{g_{i+1,j+1}(t,q) + g_{i+1,j-1}(t,q) - 2g_{i+1,j}(t,q)}{(\Delta q)^2} \end{aligned} \qquad \textbf{(65)}$$

In order to solve the PDE from equation 36 (under the risk-adjusted process) we have to impose the restrictions from formula 65 on every node (i,j) - that is the approximations of the partial derivatives have to satisfy the partial differential equation exactly. By combining the last equation with formula 65 we find the following approximated value for g(t,q) in node (i,j):

$$g(t,q) = \frac{1}{1 + [q_j + \psi(t_i)]^2\Delta t}[p_u g_{i+1,j+1}(t,q) + p_m g_{i+1,j}(t,q) + p_d g_{i+1,j-1}(t,q)]$$

where

$$\begin{aligned} p_u &= \frac{1}{2}\left[\frac{\sigma^2}{(\Delta q)^2} \mp \kappa j\right]\Delta t \\ p_m &= \left[\frac{1}{\Delta t} - \frac{\sigma^2}{(\Delta q)^2}\right]\Delta t \\ p_d &= \frac{1}{2}\left[\frac{\sigma^2}{(\Delta q)^2} \pm \kappa j\right]\Delta t \end{aligned} \qquad \textbf{(66)}$$

Along the lines of Hull and White (1994) we impose[18] the following relationship $\Delta q = \sigma\sqrt{3\Delta t}$, hence we can rewrite p_u,p_m and p_d as[19]:

$$\begin{aligned} p_u &= \frac{1}{6} + \frac{1}{2}\kappa^* j\Delta t \\ p_m &= \frac{2}{3} \\ p_d &= \frac{1}{6} - \frac{1}{2}\kappa^* j\Delta t \end{aligned} \qquad \textbf{(67)}$$

[18] See Hull and White (1994) for the motivation.

[19] It should here be stressed that $\kappa^*\Delta t$ is defined as $e^{-\kappa\Delta t} - 1$.

From these equations it follows that the numbers p_u,p_m and p_d are positive and sum to 1 if $-\frac{2}{3}\frac{1}{\kappa\Delta t} < j < \frac{2}{3}\frac{1}{\kappa\Delta t}$, which allows us to conclude that they can be recognized as bra probabilities. In order to prevent the probabilities from becoming negative we cannot use a lattice that is arbitrarily large, more precisely, we have to change branching process at some level j_{shift}.

We will therefore employ the following rule for the change of branching process: at some level $j^+ < \frac{2}{3}\frac{1}{\kappa\Delta t}$ we will from node (i,j) jump to either node (i+1,j), (i+1,j-1) or (i+1,j-2) that is the grid will be bounded at j^+ - where we will denote this the falling branching process. We will also at some level $j^- > -\frac{2}{3}\frac{1}{\kappa\Delta t}$ jump from node (i,j) to either node (i+1,j+2), (i+1,j+1) or (i+1,j) that is the grid will be bounded at j^- - where we will denote this the rising branching process.

For j respectively equal to -1 or +1 we find that the probabilities in the falling and the rising branching process are defined as:

$$
\begin{aligned}
&\textit{TheFallingBranchingProcess}\\
&p_u = \frac{7}{6} + \frac{3}{2}\kappa^* j\\
&p_m = -\frac{1}{3} - 2\kappa^* j\\
&p_d = \frac{1}{6} + \frac{1}{2}\kappa^* j\Delta t\\
&\textit{TheRisingBranchingProcess}\\
&p_u = \frac{1}{6} - \frac{1}{2}\kappa^* j\Delta t\\
&p_m = -\frac{1}{3} + 2\kappa^* j\Delta t\\
&p_d = \frac{7}{6} - \frac{3}{2}\kappa^* j
\end{aligned}
\qquad \textbf{(68)}
$$

Where the probabilities in the falling branching process are positive and sum to 1 if $\frac{1}{6}\frac{1}{\kappa\Delta t} < j^+ < \frac{1}{3}\frac{1}{\kappa\Delta t}$, and the probabilities in the rising branching process are positive

sum to 1 if[20] $-\frac{1}{3}\frac{1}{\kappa\Delta t} < j^{-} < -\frac{1}{6}\frac{1}{\kappa\Delta t}$ ..

This trinomial approach defined above is quite similar to the trinomial lattice model from Hull and White (1994). Both numerical approximations methods are equal to the first order of Δt. As mentioned earlier, the main difference is that in the trinomial model that is implemented here the spot interest rate is modelled directly using $r = [q + \psi(t)]^2$, whereas in the Hull an White trinomial model it is the Δt-period interest rate that is modelled.

This difference has the following two implications - one positive and one negative: Firstly, in the method advocated here we are able to use the analytical expression for ψ(t) (from formula 55) which means that we do not need to use forward induction in building the lattice - which clearly speeds up the calibration. Secondly, the method devised here has that undesired property that the initial term structure is not matched by construction - as is the case when using forward induction in the Hull and White trinomial approach[21]. We will for this reason prefer (if possible) to use the Hull and White approach.

For the pricing of generel path-dependent interest rate derivatives a lattice method might not be the most efficient technique[22].

For the general problem of handling path-dependency, two different kinds of method can be employed:

Monte Carlo sampling from the lattice
Monte Carlo simulation

In this paper I will not discuss how to employ efficient sampling techniques in connection with lattice based methodology, this will be left for future research.

In section 8 I will for that reason show how MC techniques can be applied to one-factor interest rate models with a time-dependent drift part[23].

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[20] In practise we define $j_{shift} = j^{+} = |j^{-}| = \left|\frac{1}{6}\frac{1}{\kappa\Delta t}\right|$ - that is we change branching pro given opportunity.

[21] However it is worth mentioning that the initial yield curve is only matched at discrete maturity-points.

[22] Hull and White (1993b) have shown that certain kinds of path-dependency can be handled in a lattice-based model. Their method is however not easy to generalise.

[23] Extending the method to the case of time-dependent diffusion parameters is however straightforward,

8: Time-dependent SDEs and Monte Carlo simulation

The main idea of the Monte Carlo method can be explained by a simple example. Consider the valuation of the expected value of a random variable X distributed according to its probability density f(x). Suppose we can, by whatever means, sample a point from the distribution of X, then a direct way to calculate the expectation of X is to independently sample a large number of points, say N points, from the distribution of X and compute the arithmetic mean of these points, as:

$$X_N = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_N}{N} \tag{69}$$

Where x_i, for i = 1,2,....,N, are the points sampled from f(x). If the second moment of X is finite, then by the law of large numbers, $X_N \to E(X)$ as $N \uparrow \infty$, with probability 1. This means that for a sufficient large number of N and any particular realisation of the set of sampled points from f(x), the arithmetic mean of these points is very close to the mean of X. In fact, the mean of X_N is the same as the mean of X, and the variance of X_N is:

$$\begin{aligned} Var(X_N) &= E\left[\left(\sum_{n=1}^{N}\frac{1}{N}x_n - E(X)\right)^2\right] \\ &= \frac{1}{N^2}\sum_{n=1}^{N}E[x_n - E(X)]^2 + \frac{1}{N^2}\sum_{1\le i<j\le N}E[(x_i - E(X))(x_j - E(X))] \end{aligned} \tag{70}$$

Since all x_i are identically and independently sampled from f(x), the correlation among these x_i is zero and the variance of x_i is equal to that of X. Therefore $Var(X_N) = Var(X)/N$. This relationship is essential to the understanding of Monte Carlo. On one hand it ensures that as the number of sample points increases, the error of the estimated expected value of X decreases. On the other hand, it shows the difficulty of the Monte Carlo method in achieving a high degree of accuracy: the standard deviation descreases only as $O\left(N^{-\frac{1}{2}}\right)$.

As the complexity of traded instruments has grown, Monte Carlo techniques have become increasingly important, a trend that is likely to continue in the future. This is particularly true in the fixed income markets, where multiple factors and path-dependency are embedded in a wide variety of new structured or mortgage-based contracts. Despite its increasing importance and wide applicability, the Monte Carlo method has received much less attention in the litterature than lattice models. One reason for this is, of course, the low order of convergence $O\left(N^{-\frac{1}{2}}\right)$ which tends to make practical usage painfully slow.

In modern finance, the prices of the basic securities (the underlying state variables) are often modelled by specifying one or more Stochastic Differential Equations (SDEs). The pricing of,

for example, an option that depends on one or more of these basic securities can, under the assumption of no arbitrage, be expressed as the expected value of its discounted payouts. Where this expectation is taken with respect to an appropriate transformation of the original probability measure.

The Monte Carlo method lends itself naturally (as shown above) to the evaluation of security prices represented as expectations. The Monte Carlo approach consist of the following steps:

> Simulate sample paths of the underlying state variable/s (underlying asset price and interest rate) over the relevant time horizon. Where the simulation is performed under the risk-neutral measure.
> Evaluate the discounted cash-flows of the security on each sample path, as determined by the structure of the security in question
> Average the discounted cash-flows over the sample paths

In principle this method computes a multi-dimensional integral - the expected value of the discounted payouts over the space of sample paths.

It might be worth pointing out that in the case of time-dependent SDEs for interest rate models we have to condition the path-sampling process - a method to solve this problem will be developed later in this section.

Before continuing, let me for a moment dwell on the pricing of contingent claims using the risk-neutral density pricing approach[24]. For a derivative asset with payoff defined by H(y(T)) and price g(y,t,T) we have:

$$g(y,t,T) = P(t,T)\int_{-\infty}^{\infty} H(y(T))p(y(T))dy(T) \qquad \textbf{(71)}$$

Where p(y(T)) is the risk-neutral density function. Formula 71 represents the price at time t of a contingent claim that has a payoff defined by H(y(T)) at time T which is a function of the process that governs the underlying state-variable - here represented by y.

Since the risk-neutral density pricing approach expresses all asset prices as the discounted value of their expected payoff formula 71 is identical to formula 5. Equation 71 (and therefore equation 5) is very powerful due to the fact that any derivative security may be priced by this approach[25].

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[24] The risk-neutral density function p(y(T)) is also known as Green's function or the fundamental PDE solution. p(y(T)) also satisfies the Kolmogorov forward (or Fokker-Planck) equation with a boundary condition given by the Dirac delta-function, see Cox and Miller (1965 chapter 5) and sections 2 and 3 in this paer.

[25] This is also true in the case of multivariate contingent claims. The only difference between valuing multivariate and univariate claims is that the payoff function for the multivariate claims involves the prices of multiple underlying assets, and the expectation is taken over all the underlying asset prices. The solution to formula

The central point in the valuation of derivative securities is the expression for the risk-neutal density function p(y(T)) - which is implied from the SDE/s for the underlying state variable/s.

In this paper we have considered two different assumptions about the spot-rate process - which both turned out to imply a normal risk-neutral density function[26].

In connection with the pricing of stock-options the most common assumption is to assume a lognormal risk-neutral density for the stock price - that is adopt the assumption from Black and Scholes (1973).

The normal risk-neutral density assumption for interest rate models and the lognormal risk-neutral density assumption for stock models is not neccesarily the appropriate assumption to use, as departures from both the normality and lognormality are well documented in many financial time-series, see for example Bollerslev, Engle and Nelson (1993). These assumptions are primarily used for practical reasons because of their great tractability.

An alternative approach could therefore be to use a non-parametric approach in the determintion of the density function. One approach could be to use the kernel estimator to determine the empirical density function, for additional information see Scott (1992 chapter 6), Ait-Sahalia (1996a,1996b) and Campbell, Lo and Mackinlay (1997 chapter 12 section 12.3)[27].

An important issue in connection with Monte Carlo simulation is how to improve the efficiency without the need to make N (the number of paths) unreasonable large - and therefore computation time too slow for practical purposes.

Using Monte Carlo simulation we can express the value of g(y,t,T) as:

$$g(y,t,T) \approx \frac{1}{N}\sum_{i=1}^{N} g_i(K) \qquad \textbf{(72)}$$

Where K is the dimension of the problem - that is the number of time-steps used in the discrete time approximation of the SDE. $g_i(K)$ is the discounted value of the payoff conditioned on the ith sample path. Furthermore we have that $g_i(K)$ is derived from the

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71 is now found by computing a multi-dimensional integral.

[26] Assuming a normal risk-neutral density function for the interest rate model does not neccesarly imply that interest rates can become negative - it does imply that the state variable can become negative but the probability of negative interest rates depends on the relationship between the state variable and the spot-rate (se section 2).

[27] A number of techniques for estimating state prices have been proposed in the context of option pricing, see for example Jackwerth and Rubenstein (1996) and Rosenberg (1997). Both these approaches imply the risk-neutral density from quoted options prices. Other related works are Andersen and Brotherton-Ratcliffe (1998) which utilises a Crank-Nicholson finite-difference scheme and Dupire (1994) who uses the explicit finite-difference scheme.

random variables, as:

$$g_i(K) = S(z_1(i), z_2(i), ..., z_K(i)) \quad \textbf{(73)}$$

Where $z_1(i), z_2(i),...,z_K(i)$ is an independent sequence of standard Gaussian variables and $S: \Re^k \rightarrow \Re$ is a well-behaved function which is a function of the used discretization scheme[28].

To enhance the efficiency of the Monte Carlo simulation, two routes can be taken (or combinations hereof):

Reduce the variance of the sampled variance
Overcome the $O\left(N^{-\frac{1}{2}}\right)$ barrier

Different variance reduction techniques have been developed to increase precision. Two of the classical variance reduction methods are the control variate approach and the antithetic variate method. Recently, moment matching, importance sampling, empirical martingale simulation techniques, the Brownian-Bridge method, conditional Monte Carlo methods and variance reduction by Girsanovs transformation have been introduced in finance litterature, see for example Boyle, Broadie and Glasserman (1995), Duan and Simonato (1998), Caflisch, Morokoff and Owen (1997) and Schoenmakers and Heemink (1997)[29].

To overcome the $O\left(N^{-\frac{1}{2}}\right)$ barrier a Quasi-Monte Carlo simulation can be employed. This method uses a determinstic sequence that is more uniform than random[30].

This can best be explained by rewriting equation 73 as:

$$g_i(K) = S(z_1(i), z_2(i), ..., z_K(i)) = Q(u_1(i), u_2(i), ..., u_K(i)) \quad \textbf{(74)}$$

Where $u_1(i), u_2(i),...,u_K(i)$ is an independent sequence of standard uniform U(0,1)-variables

[28] By this we mean - which kind of discretization has been used for the SDE - this will be explained in more detail later in this section.

[29] In Appendix G, I will briefly explain the different variance reduction techniques used later in this paper.

[30] Using the word random - does not mean truly random as the "random" numbers are generated by a deterministic algorithm and are described as pseudorandom numbers. The pseudorandom number generator used in the examples in section 9 is an example of the linear congruential kind introduced in 1948 by the mathematician D. H. Lehmer. It depends on two numbers, 16.807 and 2.147.483.647. The first is 7^5, a primitive root of the second, which is Eulers prime, 2^{31}. We have repeated the results obtained in this paper using the Mersenne twister (see http://en.wikipedia.org/wiki/Mersenne_twister), but the results we get are nearly identical.

and Q: $\Re^k \rightarrow \Re$ is a well-behaved function[31].

Using 74 the expectation of g(y,t,T) can be written as an integral over the k-dimensional hypercube:

$$g(y,t,T) = E_t[g(K)] = \int_{[0;1]^K} Q(x_1, x_2, ..., x_K)dx_1dx_2....dx_K \approx \frac{1}{N}\sum_{i=1}^{N} Q(x_i) \qquad (75)$$

Formula 75 is identical to formula 72 except that we have not specified which sampling scheme has been used. If the sampling scheme is Monte Carlo (that is, based on a pseudorandom number generator) we know that the expected error in formula 75 is independent of the dimension k and proportional to $N^{-\frac{1}{2}}$. To improve this convergence criterium, several deterministic sampling algorithms have been suggested instead of Monte Carlo simulation.

Specific algorithms for generating quasi-random numbers has been suggested, for example Sobol (1967), Halton (1960) and Faure (1982), see Press, Teukolsky, Vetterling and Flannery 1992 (chapter 7) for a good description of these methods.

Due to Niederreiter (1992 page 20) one can show (the Koksma-Hlawka Inequality) that under technical conditions on the function Q, the Sobol and Halton sequences generate errors which decrease with N^{-1} - at least as $\frac{(\log N)^K}{N}$ [32].

For small dimensions the discrepancy for quasi-random number generators appears to be $O(N^{-1})$, ignoring logarithmetic factors, for all N. For large dimensions, the discrepancy behaves initially like $O(N^{-\frac{1}{2}})$ - as for the random number sequence - converging only to $O(N^{-1})$ for very large values of N.

The value of N appears to grow exponentially with the dimensions - which implies that in high dimensions - unless N is extremely high - quasi-random number sequences are no more

[31] The transformation from Gaussian to uniform variates can be accomplished by inverting the cumulative Gaussian distribution or through the inverse of the Box-Mueller transformation.

[32] Which is smaller than the error for Monte Carlo simulation because the error bound $O\left(\frac{(\log N)^K}{N}\right)$ is smaller than $O\left(N^{-\frac{1}{2}}\right)$ as $N \rightarrow \infty$.

uniform than random number sequences. Thus the Koksma-Hlawka bound does not imply that quasi-random number sequences are more efficient than random number sequences for moderate values of N and large dimensions K.

It is usually the case that Quasi-Monte Carlo methods are superior to Monte Carlo simulation (or at least no worse), see for example Paskov (1996), Acworth, Broadie and Glasserman (1997) and Caflisch, Morokoff and Owen (1997) - even in high-dimensionel situations. Furthermore, the results from Paskov (1996) indicate that the Sobol sequence frequently outperforms other kinds of quasi-random number sequences.

So far we have discussed varies aspects in connection with Monte Carlo simulation and mentioned ways to improve the efficiency. Let us now turn our attention to the SDE we wish to simulate:

$$\begin{gathered} dy = (\varphi(t) - \kappa y)dt + \sigma dW \\ where \\ r = F(t,y) \end{gathered} \qquad \textbf{(76)}$$

which can be recognized as being identical to formula 1 - for the diffusion coefficient defined as $\sigma(y,t) = \sigma$ and the drift coefficient defined as $\mu(\varphi(t),y,t) = \varphi(t) - \kappa y$.

For the purpose of simulation let us instead consider the following diffusion process (SDE):

$$\begin{gathered} dx = -\kappa x dt + \sigma dW \quad for\ x_0 = 0 \\ where \\ y = x + \varphi(t) \end{gathered} \qquad \textbf{(77)}$$

Generally speaking, a diffusion process is an arbitrary strong Markov process with continous sample paths. In our framework, a diffusion process is given as a strong solution[33] of an SDE driven by the underlying Brownian motion W.

With respect to discrete time approximations of continous time processes, it is important to distinguish between strong and weak convergence. The weak convergence criterion does not (as the strong convergence) require a pathwise approximation of the Ito-process, but only an approximation of the probability distribution (see Kloeden and Platen (1995 chapter 4, section 9.6 and 9.7).

In Kloeden and Platen (1995 parts V and VI) it is shown that the strong and weak convergence criteria leads to the development of different sampling schemes which are only efficient with respect to one of the two criteria. This means that a given sample scheme usually has different orders of convergence with respect to the two criteria.

[33] A strong solution to formula 77 requires that the SDE satisfies the Lipschitz condition, the linear growth bound and initial value condition. It is furthermore required that the coefficient in the SDE is continuous - with imply measurability, see Kloeden and Platen (1995, sections 4.5 and 4.6).

We will here consider three different strong convergence sampling schemes, the Euler scheme, the Milstein scheme and the 1.5 strong Taylor scheme. Futhermore we will consider two weak convergence sampling schemes, the Euler scheme and the 2. order Milstein scheme.

Before I go through the three different sampling schemes let me make the following definitions.

We will here consider a time-discretization τ of the time interval [t,T]:

$$t = \tau_1 < \tau_1 < \tau_2 <....< \tau_k <....< t_K = T \qquad \textbf{(78)}$$

Where K denotes the number of discretization intervals $\Delta_k = \tau_{k+1} - \tau_k$. As we only consider the case of equidistant discretization points, we have that $\delta = \Delta_k = \frac{T - t}{K}$ for all k.

8.1 The Euler scheme

The Euler discretization scheme is given by:

$$x_{k+1} = x_k - \kappa x_k \delta + \sigma \Delta W \qquad \textbf{(79)}$$

Where $\Delta W = W_{\tau_{k+1}} - W_{\tau_k}$ is the N(0,δ) distributed increments of the Wiener process W o $[\tau_k, \tau_{k+1}]$.

This discretization scheme represents the simplest strong Taylor approximation and attains the order or convergence 0.5, see Kloeden and Platen (1995 section 10.2).

Furthermore, we have that the Euler weak approximation is of a form identical to equation 79, however, here the order of convergence is 1, see Kloeden and Platen (1995 section 14.1).

8.2 The Milstein scheme - strong 1. order Taylor approximation

The Milstein discretization scheme is defined as:

$$x_{k+1} = x_k - \kappa x_k \delta + \sigma \Delta W + \frac{1}{2}\sigma^2((\Delta W)^2 - \delta) \qquad \textbf{(80)}$$

which (as seen from the equation) is obtained by adding the term $\frac{1}{2}((\Delta W)^2 - \delta)$ to the Euler scheme.

In Kloeden and Platen (1995 section 10.3) it is shown that the Milstein scheme converges strongly with order 1.

8.3 The order 1.5 strong Taylor scheme

The order 1.5 strong Taylor scheme can in abstract form be written as[34]:

$$x_{k+1} = x_k + \mu\delta + \sigma\Delta W + \frac{1}{2}\sigma\sigma_x((\Delta W)^2 - \delta) + \mu_x\sigma\Delta Z + \frac{1}{2}\left(\mu\mu_x + \frac{1}{2}\mu_{xx}\sigma^2\right)\delta^2 + \left(\mu\sigma_x + \frac{1}{2}\mu_{xx}\sigma^2\right)(\Delta W\delta - \Delta Z) + \frac{1}{2}\sigma\left(\sigma\sigma_x + (\sigma_x)^2\right)\left(\frac{1}{3}(\Delta W)^2 - \delta\right)\Delta W \quad \textbf{(81)}$$

where

$$\mu = -\kappa x_k$$

Where the footsigns (x) represent partial derivatives with respect to x. The additional random variable ΔZ is defined by the following double integral:

$$\Delta Z = \int_{\tau_k}^{\tau_k+1}\int_{\tau_k}^{s_2} dW_{s_1} ds_2 \quad \textbf{(82)}$$

where we have that the random variable ΔZ is distributed as $N\left(0, \frac{1}{3}\delta^3\right)$. Furthermore we have that the covariance $E[\Delta Z \Delta W] = \frac{1}{2}\delta^2$.

For our particular SDE - given by formula 77 - we can simplify formula 81and get the following expression for the order 1.5 strong Taylor scheme:

$$x_{k+1} = x_k - \kappa x_k\delta + \sigma\Delta W - \kappa\sigma\Delta Z + \frac{1}{2}\kappa^2 x_k\delta^2 \quad \textbf{(83)}$$

8.4 The order 2 weak Milstein scheme

The order 2 weak Milstein scheme can in abstract form be written as[35]:

$$x_{k+1} = x_k + \mu\delta + \sigma\Delta W + \frac{1}{2}\sigma\sigma_x((\Delta W)^2 - \delta) + \mu_x\sigma\Delta Z + \frac{1}{2}\left(\mu\mu_x + \frac{1}{2}\mu_{xx}\sigma^2\right)\delta^2 + \left(\mu\sigma_x + \frac{1}{2}\mu_{xx}\sigma^2\right)(\Delta W\delta - \Delta Z) \quad \textbf{(84)}$$

where

$$\mu = -\kappa x_k$$

[34] The order of convergence is shown in Kloeden and Platen (1995 section 10.4).

[35] The order of convergence is shown in Kloeden and Platen (1995 section 14.2)

Where the footsigns (x) represent partial derivatives with respect to x. The variable ΔZ is defined in formula 82. Comparing formula 84 to formula 81 it is easily seen that the order 2 weak Taylor approximation is equal to the order 1.5 strong Taylor approximation if the last term in line 2 in equation 81 is removed[36].

For our particular SDE, we actually have that the order 1.5 strong Taylor approximation is equal to the order 2 weak Taylor approximation - that is equal to formula 83.

Before deciding to use either the higher order strong-or weak-discrete approximations[37] it is important to clarify the aim of a simulation. The following question should be answered:

Is a good pathwise approximation of the Ito process required or is the approximation of some functional of the Ito process the real objective?

In many cases, also the examples considered in section 9, the weak discrete-time approximation is the appropriate one as we in many cases are just interested in the probability distribution and its moments at given times T[38].

An interesting point to emphasize at this point is that equation 77 is actually a fairly general formulation[39] for one-factor interest rate models with a time-dependent drift term as we by making appropriate choices for F(t,y) (as mentioned in the beginning of section 2) get the Extended Vasicek model from Hull and White (1990) ($F(t,y) = y$), Black and Karisinski (1991) ($F(t,y) = e^y$), the Quadratic interest rate model ($F(t,y) = y^2$) and the Cox, Ingersoll and Ross (1985) model for $F(t,y) = \sqrt{y}$.

From the points discussed so far it is possible to conclude that the efficiency of simulating the diffusion process defined by the SDE in formula 77 is a function of the following:

The discretization scheme
The Monte Carlo simulation procedure

In section 9 we will for the Extended Vasicek model and the Quadratic interest model for the

[36] Higher order strong-and weak discretization schemes can of course also be derived but for strong schemes above 1.5 and weak schemes above 2 the expressions get pretty complicated. As the expansions to higher order approximations of the SDE leads to non-trivial multiple stochastic integrals. An atractive alternative to the truncated Taylor series expansion - at least in the case of weak discretizations shemes - is to use Richardson or Romberg extrapolation, see Kloeden and Platen (1995 page 285).

[37] We are here referring to discrete schemes of higher order than Euler schemes.

[38] A situation where we would be interested in a pathwise approximation of the Ito process is in the case of filtering.

[39] Expanding to the case of more complex drifts specifications or for that sake diffusion coefficients is relatively straightforward. This issue will however not be pursued here - as the formulation chosen here covers the typical assumptions used in the literature.

case of a call and put option on a zero-coupon bond analyse some combination of discretization schemes and Monte Carlo methods. Furthermore we will as comparison price the same instruments analytically and by using a lattice approach.

Before that we will in the rest of this section show a method to condition the path-sampling process in order to ensure that the initial yield-curve is matched.

8.5 Constrained Monte Carlo simulation

Our basic SDE is (as shown above) defined as:

$$dx = -\kappa x dt + \sigma dW \quad for\ x_0 = 0 \tag{85}$$

Let us assume that we have split the time interval [t,T] into K discretization intervals of an equal size, that is $\delta = \frac{T - t}{K}$. Furthermore we assume that we have performed N simulations of the SDE.

That is we have an NxK matrix of simulated values for x - which we will denote the diffusion-matrix.

It might be worth mentioning that the procedure described below is independent of the chosen discretization scheme and the Monte Carlo method employed in the construction of the diffusion matrix - which of course makes the procedure extremely general and quite flexible.

Given is also the yield-curve (or equally the discount function), which means that we know the price P(t,kδ), for all k = [1,K] and where P(t,kδ) = 1 for t = kδ.

We now wish to construct a new diffusion matrix (RD-matrix) for the spot-rate, F(t,y) = r, where y = x + φ(t). That is we now introduce the correct, time-varying drift. To do this we will change the values of each cell in the diffusion-matrix. The change of the diffusion-matrix will however be equal for all k's as φ(kδ) is only time-dependent and not path-dependent.

The method - which is a forward-induction principle - can be described as[40]:

Step 1: Column 1 in the RD-matrix is known from the yield-curve and is defined as $r_1 = \frac{-\ln P(t,\delta)}{\delta}$. Where we therefore have that $\varphi(1) = r_1$.

Step 2: We will now introduce the concept of Arreu-Debreu prices.

[40] Forward-induction in connection with Monte Carlo and time-dependent spot-interest rate models might have been around for some time but until now this technique has not (as far as I know) been described in the literature.

An Arreu-Debreu is a security that has a payoff of 1 at time kδ in state n = [1,N]. In general we have that the vector of Arreu-Debreu prices at time kδ can be written as:

$$Q(t,k\delta) = e^{-\tilde{r}_k\delta}Q(t,(k-1)\delta)$$

where

$$\tilde{r}_k = r_{k-1} + x_{k-1}\delta$$

We have that Q(t,kδ) = 1 for t = kδ, and $\tilde{r}_k$ represent the forecasted vector of spot-rates for the period kδ - (k+1)δ.

Step 3: The true vector of spot-rates can now be derived from the following equation:

$$P(t,k\delta) = Q(t,k\delta)e^{-F(t,\tilde{r}_k + \varphi(k\delta))\delta} = \frac{1}{N}\sum_{n=1}^{N} q_n(t,k\delta)e^{-F(t,\tilde{r}_k(n) + \varphi(k\delta))\delta}$$

In some case we can solve this expression directly for φ(kδ) - this is for example the case in the Extended Vasicek. When this is not the case we need to estimate φ(kδ) using a minimization algorithm - this can however (in most cases) be done in 1-2 iterations.

In some cases it is even possible to derive an analytical expression for φ(t). In these cases (as pointed out in section 7) we did not need to use forward-induction in calibrating the lattice as we could determine φ(t) analytically. This is however only possible for the Quadratic interest rate model, see formula 55 and for the extended Vasicek model, see Appendix E formula 6.

This approach could also be utilized here. If that was the case the same implications mentioned in connection with the lattice method in the end section 7 would apply here.

Step 4: Repeating Step 2-3 for all k > 1 builds the diffusion RD-matrix. Where the following equation by construction is now true for all k = [1,K]:

$$P(t,k\delta) = \frac{e^{-[RD_1 + RD_2 + \ldots RD_k]\delta}}{N}$$

Where RD_k is the k'th row in the diffusion RD-matrix with a length equal to N. This equation just states that the average price from the sampled process is equal to the corresponding market price for all k = [1,K].

From the description above it follows that the method designed here - for the case of path-dependent simulation of the interest rate process - is very similar in spirit to the Hull and White (1994) lattice-building procedure.

9: Some pricing examples

In this section we will calculate European options-prices for both the Extended Vasicek model

and the Quadratic Interest rate model.

In this connection we assume that the yield-curve is defined as: $0.08 - 0.05e^{-0.18(T-t)}$
Given this term structure we will now price 3-year call-and put-options where the underlying security is an 8-year zero-coupon bond.

From the yield-curve we can deduce that the forward-price P(3,8) = 67.53. Therefore we have chosen to calculate the option-prices for the following selection of strike-prices: 52.53, 62,53, 67.53, 72.53 and 82.53. For both models we have assumed that $\kappa = 0.05$ and $\sigma = 0.01$.

In table 1 we have shown the analytical values of the options for both the Extended Vasicek model and the Quadratic Interest rate model:

Table 1: Analytical Option Prices

	Call option prices		**Put option prices**	
Exercise Prices	**The Extended Vasicek Model**	**The Quadratic Interest Rate Model**	**The Extended Vasicek Model**	**The Quadratic Interest Rate Model**
52,53	*12,877*	*12,877*	*0,000*	*0,000*
62,53	*4,576*	*4,311*	*0,283*	*0,019*
67,53	*1,647*	*0,847*	*1,647*	*0,847*
72,53	*0,354*	*0,015*	*4,647*	*4,307*
82,53	*0,003*	*0,000*	*12,881*	*12,877*

In the case of the Extended Vasicek model we have also calculated the option prices by performing a simulation under the T^F-adjusted probability measure - where T^F symbolises the expiry-date of the option (that is $T^F = 3$). More precisely we have transformed the Extended Vasicek model into the Heath, Jarrow and Morton framework - which mean that it is possible to write the forward-price $P(T^F,T)$ as:

$$P(T^F,T) = \frac{P(0,T)}{P(0,T^F)} \exp\left[-\frac{1}{2}\int_0^{T^F} [\sigma_P(s,T) - \sigma_P(s,T^F)]^2 ds - \int_0^{T^F} [\sigma_P(s,T) - \sigma_P(s,T^F)] dW^{Q_{TF}}(s) \right] \quad (86)$$

Where $\sigma_p(t,T)$ is the bond-price volatility - which is related to the forward-rate volatility through the following relationship:

$$\sigma_P(t,T) = \int_t^T \sigma^F(t,v)dv = \frac{\sigma}{\kappa}[1 - e^{-\kappa(T - t)}] \quad \textbf{(87)}$$

What we have done is simulated the forward-price $P(T^F,T)$ using equation 86. We have employed the following Monte Carlo simulation methods: Crude Monte Carlo, Antithetic Monte Carlo, Stratified Sampling and Empirical Martingale Simulation (EMS)[41].

We have performed B batches of N simulations, for B = 100 and N = 1000. Provided that B is sufficiently large, the Central Limit Theorem implies that the batch error is approximately Gaussian. The sampled variance can under this assumption be calculated as:

$$Var_B = \frac{1}{B(B - 1)}\left(B\sum_{b=1}^{B} \hat{C}_b^2 - \left(\sum_{b=1}^{B} \hat{C}_b \right)^2 \right) \quad \textbf{(88)}$$

Where $\hat{C}_b$ is the estimated/simulated option-price for the b'th batch-run.

Table 2: Simulated Call-option prices in the Extended Vasicek Model

Excercise Price	**Crude Monte Carlo**	**Std. Error[42]**	**Antithetic Monte Carlo**	**Std. Error**	**Stratified Sampling**	**Std. Error**	**Empirical Monte Carlo Simulation**	**Std. Error**
52,53	*12,864*	*0,127*	*12,877*	*0,006*	*12,878*	*0,002*	*12,877*	*0,001*
62,53	*4,572*	*0,125*	*4,573*	*0,025*	*4,576*	*0,002*	*4,578*	*0,023*
67,53	*1,647*	*0,083*	*1,648*	*0,041*	*1,647*	*0,001*	*1,649*	*0,033*
72,53	*0,350*	*0,033*	*0,350*	*0,023*	*0,354*	*0,002*	*0,351*	*0,028*
82,53	*0,003*	*0,003*	*0,003*	*0,002*	*0,003*	*0,001*	*0,003*	*0,003*

[41] In Appendix G these methods are briefly explained.

[42] Std. Error is calculated as the square root of the sampled variance specified in formula 88.

Table 3: Simulated Put-option prices in the Extended Vasicek Model

Excercise Price	Crude Monte Carlo	Std. Error	Antithetic Monte Carlo	Std. Error	Stratified Sampling	Std. Error	Empirical Monte Carlo Simulation	Std. Error
52,53	*0,000*	*0,001*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
62,53	*0,284*	*0,030*	*0,284*	*0,019*	*0,284*	*0,001*	*0,284*	*0,025*
67,53	*1,655*	*0,071*	*1,650*	*0,037*	*1,647*	*0,001*	*1,648*	*0,039*
72,53	*4,660*	*0,106*	*4,645*	*0,018*	*4,647*	*0,001*	*4,647*	*0,026*
82,53	*12,869*	*0,142*	*12,880*	*0,006*	*12,880*	*0,001*	*12,880*	*0,003*

From these two tables it can be concluded (as expected) that Crude Monte Carlo simulation performs worst. Furthermore it follows that of the variance reduction methods employed here Stratified sampling is the most efficient method, and EMS performs marginally better than Antithetic. It is usually the case that the Antithetic variates technique is the variance reduction method which is least efficient.

The reason why the EMS variance reduction method does not perform better than it does can be explained as follows: The main idea behind the EMS method is namely to adjust the sampling in such a way that the average price of the underlying instrument is equal to the expected price (forward-price) of the underlying instrument[43]. As the sampling using formula 86 is performed under the T^F-adjusted probability measure - which in a sense is to move the discounting outside the expectation - we have therefore already lowered the uncertainty in the sampling procedure, as we have[44]:

[43] Se Appendix G.

[44] The terms in the first expectation in formula 89 can be recognized as the Radon-Nikodym derivative (the likelihood ratio process) which under technical conditions for $\sigma_P(t,T)$ accordingly to Girsanov Theorem relates the exisisting probability measure Q to a new equivalent probability measure (in this example Q_{TF}), see Musiela and Rutkowski (1997 Appendix B.2).

$$E^{Q}\left[\exp\left[-\frac{1}{2}\int_{0}^{T^{F}}\sigma_{P}(s,T^{F})^{2}ds - \int_{0}^{T^{F}}\sigma_{P}(s,T^{F})dW^{Q}(s)\right]\right] = 1$$

or equivalently **(89)**

$$E^{Q_{TF}}\left[\exp\left[-\frac{1}{2}\int_{0}^{T^{F}}[\sigma_{P}(s,T) - \sigma_{P}(s,T^{F})]^{2}ds - \int_{0}^{T^{F}}[\sigma_{P}(s,T) - \sigma_{P}(s,T^{F})]dW^{Q_{TF}}(s)\right]\right] = 1$$

From this it follows that the technique employed using formula 86 in the simulation procedure is equal to the measure transformation method developed by Andersen (1995) in connection with Monte Carlo simulation of one-factor non-linear time-homogenous interest rate models.
Before we calculate the prices using Monte Carlo simulation for both the call-and put-options for the two different weak discretization schemes for the SDE from section 8 we will first calculate the option-prices using a lattice-model.

We will if nothing else is mentioned (if possible[45]) be using the Hull and White (1994) lattice approach.

Table 4[46]: Call-and Put-option prices in the Extended Vasicek Model using the Hull and White (1994) Lattice approach

Call-Option prices					
Number of time-steps	**8**	**16**	**32**	**64**	**128**
52,53	*12,880*	*12,879*	*12,878*	*12,878*	*12,878*
62,53	*4,603*	*4,579*	*4,574*	*4,579*	*4,576*

[45] See the discussion at the beginning of section 7 for cases where that is not possible. It is however of importance to mention here that the probability of not being able to use the Hull and White procedure to match the initial yield-curve - is larger in lattice-based models than in Monte Carlo based models. The reason for is because the interest-space that a lattice-based method spans is larger than the one covered in a Monte Carlo procedure. In contrast to this the interest-space that a Monte Carlo method covers is finer than the one covered by the lattice-based methodology. The conclusion now follows as the probability - of not being able to use the Hull and White procedure - is bigger the lower the interest rates, see formula 54.

[46] It is here worth noting that we in the calculation of the option-prices using the lattice approach have made use of the fact that we have a closed form solution for the bond-price, for further details see Appendix H.

67,53	*1,631*	*1,645*	*1,649*	*1,650*	*1,649*
72,53	*0,383*	*0,362*	*0,357*	*0,358*	*0,356*
82,53	*0,004*	*0,003*	*0,003*	*0,003*	*0,003*
Put-Option prices					
52,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
62,53	*0,308*	*0,285*	*0,281*	*0,286*	*0,284*
67,53	*1,628*	*1,643*	*1,649*	*1,650*	*1,649*
72,53	*4,673*	*4,653*	*4,648*	*4,650*	*4,648*
82,53	*12,880*	*12,880*	*12,880*	*12,880*	*12,880*

As is obvious from table 4, the convergence is very fast using a lattice-based methodology.

We have in table 5 performed the same calculations for the Quadratic interest rate model. As was the case for the Extended Vasicek model we have also here used the fact that it is possible to get a (semi) closed form solution for the bond-price[47].

Table 5: Call-and Put-option prices in the Quadratic Interest Rate Model using the Hull and White (1994) Lattice approach

Call-Option prices					
Number of time-steps	**8**	**16**	**32**	**64**	**128**
52,53	*12,879*	*12,878*	*12,878*	*12,877*	*12,877*
62,53	*4,313*	*4,313*	*4,311*	*4,312*	*4,311*
67,53	*0,841*	*0,845*	*0,846*	*0,847*	*0,847*
72,53	*0,015*	*0,017*	*0,015*	*0,015*	*0,015*
82,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
Put-Option prices					
52,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
62,53	*0,019*	*0,020*	*0,019*	*0,020*	*0,019*

[47] For elaboration see Appendix H.

67,53	*0,840*	*0,844*	*0,846*	*0,847*	*0,847*
72,53	*4,306*	*4,308*	*4,308*	*4,307*	*4,307*
82,53	*12,876*	*12,877*	*12,877*	*12,877*	*12,877*

Again we see, as was the case for the Extended Vaiscek model, a fast convergence to the true result.

As our first step we have now in comparison calculated the call-and put-option prices for both the Extended Vasicek model and the Quadratic interest rate model using crude Monte Carlo simulation for the following number of steps: k=[8,16,32,64,128]. We have performed the calculation for the two different weak discretizations scheme for the SDE which was shown in section 8.

We have performed B batches of N simulations, for B = 100 and N = 1000, and have used the constrained MC-technique presented in section 8.4 in order to ensure that the initial yield-curve is matched for all k for each batch. Furthermore we have - as when using the lattice-based approach - used the knowledge that we can obtain a (semi) closed form solution for the bond-price.

The calculated put-option prices for the Extended Vasicek model are shown in table 6 and for the Quadratic interest rate model in table 7. The calculated call-option prices for the two models are shown in Appendix I.

Table 6: Put-option prices in the Extended Vasicek Model using Constrained Crude Monte Carlo Simulation

	K=8	**Std. Error**	**K=16**	**Std. Error**	**K=32**	**Std. Error**	**K=64**	**Std. Error**	**K=128**	**Std. Error**
Euler Discretization scheme - order 1 weak Taylor approximation										
52,53	*0.000*	*0.001*	*0.000*	*0.000*	*0.000*	*0.001*	*0.000*	*0.001*	*0.000*	*0.001*
62,53	*0.278*	*0.019*	*0.280*	*0.025*	*0.285*	*0.023*	*0.276*	*0.023*	*0.275*	*0.021*
67,53	*1.552*	*0.033*	*1.609*	*0.043*	*1.627*	*0.040*	*1.640*	*0.036*	*1.637*	*0.036*
72,53	*4.466*	*0.025*	*4.569*	*0.031*	*4.603*	*0.020*	*4.630*	*0.021*	*4.632*	*0.020*
82,53	*12.639*	*0.008*	*12.765*	*0.006*	*12.825*	*0.005*	*12.854*	*0.006*	*12.866*	*0.008*
Milstein Discretization scheme - order 2 weak Taylor approximation										
52,53	*0.000*	*0.001*	*0.000*	*0.001*	*0.000*	*0.001*	*0.000*	*0.000*	*0.000*	*0.000*
62,53	*0.265*	*0.017*	*0.271*	*0.026*	*0.283*	*0.022*	*0.284*	*0.016*	*0.278*	*0.021*
67,53	*1.572*	*0.041*	*1.603*	*0.042*	*1.624*	*0.039*	*1.633*	*0.041*	*1.647*	*0.033*

72,53	*4.465*	*0.031*	*4.553*	*0.027*	*4.601*	*0.030*	*4.628*	*0.028*	*4.635*	*0.025*
82,53	*12.642*	*0.008*	*12.761*	*0.005*	*12.825*	*0.008*	*12.852*	*0.007*	*12.865*	*0.006*

Table 7: Put-option prices in the Quadratic Interest Rate Model using Constrained Crude Monte Carlo Simulation

	K=8	**Std. Error**	**K=16**	**Std. Error**	**K=32**	**Std. Error**	**K=64**	**Std. Error**	**K=128**	**Std. Error**
Euler Discretization scheme - order 1 weak Taylor approximation										
52,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
62,53	*0,017*	*0,006*	*0,018*	*0,006*	*0,018*	*0,005*	*0,021*	*0,005*	*0,018*	*0,004*
67,53	*0,763*	*0,024*	*0,804*	*0,016*	*0,819*	*0,020*	*0,835*	*0,022*	*0,844*	*0,022*
72,53	*4,089*	*0,005*	*4,199*	*0,004*	*4,255*	*0,004*	*4,280*	*0,005*	*4,292*	*0,004*
82,53	*12,650*	*0,002*	*12,766*	*0,002*	*12,822*	*0,002*	*12,849*	*0,002*	*12,864*	*0,002*
Milstein Discretization scheme - order 2 weak Taylor approximation										
52,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
62,53	*0,017*	*0,004*	*0,018*	*0,005*	*0,018*	*0,006*	*0,019*	*0,005*	*0,020*	*0,004*
67,53	*0,767*	*0,016*	*0,795*	*0,020*	*0,825*	*0,019*	*0,840*	*0,019*	*0,842*	*0,018*
72,53	*4,088*	*0,004*	*4,201*	*0,004*	*4,254*	*0,003*	*4,279*	*0,005*	*4,294*	*0,004*
82,53	*12,650*	*0,002*	*12,767*	*0,002*	*12,822*	*0,002*	*12,850*	*0,002*	*12,864*	*0,002*

First of all it is easy to see that the rate of convergence is much slower than in the lattice-based method.

From the put-option prices shown in tables 6 and 7 compared to the call-option prices in Appendix I, we have the following general indication:

> The convergence for put-option prices is a process which starts out with a price lower than the true price and then rises towards the true price as a function of the increasing number of steps.
> The convergence for call-option prices is a process which starts out with a price higher than the true price and then falls towards the true price as a function of the increasing number of steps.

The possible explanation for this phenomenon is that there is a tendency for the simulation

procedure[48] to calculate/estimate the highest bond-price for the lowest number of steps - that is the convergence starts out with a higher bond-price and then fall towards the true price as a function of the increasing number of steps. This explains the indications mentioned above.

Looking at the results for the two different discretizations schemes it is difficult to conclude anything in general.

In the case of simulation of spot-rate models where the spot-rate process is governed by a time-homogenous SDE, we usually have a much clearer picture - namely that using higher order schemes in the sampling process in general matters[49]. From the results we obtain for time-dependent SDE's of the particular kind used here - no clear picture is available.

Which respect to time-dependent SDE's (of the kind analysed here) we cannot in general see a more efficient convergence rate when using higher order schemes. This must be because of the procedure that forces the average price for each k = [1,K] to match the price $P(t,k\delta)$ observed in the market. Remember, namely that this is done by adjusting each element in the basic diffusion-matrix column-wise - that is as a change of drift. This procedure to match the initial yield-curve must therefore dominate over the sampling scheme.

It could be interesting to look at a couple of variance reduction schemes to see if they make the convergence rate faster. With the above results in mind it is however not clear beforehand if this will increase the efficiency.

For this purpose we will consider two different kinds of variance reduction methods[50]:

Variance reduction by using the Brownian Bridge Process
Variance reduction by Girsanovs transformation

Both these methods are very efficient, in the case of simulating spot-rate processes under the assumption of a time-homogenous SDE[51].

Because of the unclear conclusion in connection with the use of different discretizations schemes for the SDE, we have only performed the calculations using the Euler discretizations method.

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[48] This is actually also the case for the simulations I have performed for time-homogenous SDE spot-rate models.

[49] These results can be obtained by contacting the author. The time-homogenous spot-interest rate model we have tested is the Ornstein-Uhlenbeck process of the kind specified in equation 85. Similar results has been obtained by Giles (2007) - for the valuation of a range of options in the Black-Scholes model.

[50] These two variance reduction techniques is explained in Appendix G.

[51] Results indication that these variance reduction methods is efficient for spot rate models that are time-homogenous can be obtained from the author. It is however in this connection worth mentioning that by far the most efficient method of those employed was the Girsanov's transformation.

When using the Brownian Bridge method, we encountered some "strange" results. The results are shown in Appendix J. It is however worth pointing out that we have not been able to find any reason why the simulated options-prices in some cases seem to diverge instead of converge.

Our results using the method on a time-homogenous spot-rate model behaved as expected. It was clearly superior to Crude Monte Carlo simulation using the Euler scheme and slightly better than Crude Monte Carlo simulation using the 2. order weak Milstein.

The only difference between our simulations with the time-homogenous spot-rate model and the result in Appendix J, is that we in Appendix J have constrained the Monte Carlo simulation in order to match the current yield-curve. Why that should make the Brownian Bridge method diverge in some cases is not obvious, and is left for further research.

In tables 8 and 9 we have shown the result for both the Extended Vasicek model and the Quadratic interest rate model - when using Girsanovs transformation in the simulation procedure.

Table 8: Call-and Put-option prices in the Extended Vasicek Model using Constrained Girsanovs Transformation Monte Carlo Simulation

	K=8	Std. Error	K=16	Std. Error	K=32	Std. Error	K=64	Std. Error	K=128	Std. Error
Call-Option prices										
52,53	*13,120*	*0,009*	*12,997*	*0,007*	*12,939*	*0,008*	*12,907*	*0,008*	*12,894*	*0,009*
62,53	*4,802*	*0,028*	*4,692*	*0,029*	*4,626*	*0,024*	*4,601*	*0,022*	*4,592*	*0,033*
67,53	*1,800*	*0,035*	*1,715*	*0,037*	*1,674*	*0,036*	*1,669*	*0,055*	*1,657*	*0,043*
72,53	*0,423*	*0,033*	*0,373*	*0,022*	*0,368*	*0,033*	*0,365*	*0,035*	*0,359*	*0,034*
82,53	*0,006*	*0,004*	*0,004*	*0,005*	*0,003*	*0,003*	*0,003*	*0,003*	*0,002*	*0,003*
Put-Option prices										
52,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,001*	*0,000*	*0,000*	*0,000*	*0,001*
62,53	*0,271*	*0,025*	*0,272*	*0,021*	*0,284*	*0,024*	*0,277*	*0,023*	*0,280*	*0,024*
67,53	*1,553*	*0,032*	*1,601*	*0,033*	*1,621*	*0,028*	*1,649*	*0,045*	*1,632*	*0,033*
72,53	*4,464*	*0,025*	*4,567*	*0,031*	*4,602*	*0,026*	*4,623*	*0,029*	*4,639*	*0,023*
82,53	*12,639*	*0,008*	*12,761*	*0,005*	*12,822*	*0,007*	*12,850*	*0,008*	*12,865*	*0,008*

Table 9: Call-and Put-option prices in the Quadratic Interest Rate Model using Constrained Girsanovs Transformation Monte Carlo Simulation

	K=8	Std. Error	K=16	Std. Error	K=32	Std. Error	K=64	Std. Error	K=128	Std. Error
Call-Option prices										
52,53	*13,104*	*0,002*	*12,989*	*0,003*	*12,933*	*0,002*	*12,905*	*0,002*	*12,891*	*0,002*
62,53	*4,538*	*0,006*	*4,423*	*0,006*	*4,364*	*0,006*	*4,337*	*0,006*	*4,324*	*0,006*
67,53	*0,996*	*0,020*	*0,913*	*0,016*	*0,885*	*0,018*	*0,866*	*0,020*	*0,854*	*0,021*
72,53	*0,023*	*0,005*	*0,018*	*0,005*	*0,016*	*0,006*	*0,016*	*0,004*	*0,014*	*0,004*
82,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
Put-Option prices										
52,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
62,53	*0,017*	*0,004*	*0,019*	*0,005*	*0,019*	*0,005*	*0,020*	*0,005*	*0,018*	*0,005*
67,53	*0,764*	*0,016*	*0,804*	*0,019*	*0,827*	*0,020*	*0,836*	*0,018*	*0,845*	*0,017*
72,53	*4,089*	*0,004*	*4,201*	*0,004*	*4,254*	*0,004*	*4,281*	*0,004*	*4,293*	*0,004*
82,53	*12,650*	*0,002*	*12,767*	*0,002*	*12,822*	*0,002*	*12,851*	*0,002*	*12,864*	*0,002*

If the results in tables 8 and 9 are compared with respectively tables 6 and 7 and the results in Appendix I we can deduce the following:

> No clear variance reduction is obtained using Girsanovs transformation, ie no substantial fall in Std. error
> The convergence to the true option-price is not remarkably faster using Girsanonv's transformation

Employing Monte Carlo simulation for spot-rate models with a time-dependent drift that through the constraining method ensures a perfect match of the yield-curve for all steps does not seem as straightforward as in the case of time-homogenous spot-rate models. With straightforward - I mean, variance reduction is not easily obtained - at least not through conventional techniques.

Analysing the pricing of options on zero-coupon bonds using Monte Carlo and the Hull and White lattice-based approach, we can conclude the following:

> It seems as if the constraining method dominates over the variance reduction schemes (at least the ones employed here) and that it also dominates over the discretization schemes - that is a Crude Monte Carlo method using Eulers discretizations scheme does the work as well as anything else we came up with

> The Hull and White lattice-based approach is superior to the Monte Carlo simulation

Of course it is possible that by employing other tricks in connection with Monte Carlo simulation we might get better results - maybe even results that can rival the lattice-based methods. Quasi-random number sequences is a next natural step to take, or other variance reduction techniques - such as stratified sampling - but that is left for further research.

The whole idea of introducing Monte Carlo methods for time-dependent spot-rate models was to be able to price path-dependent securities in a more natural setting, as lattice-based method is not directly suitable for this kind of problem. From the results in this paper on a fairly simple instrument, we however encountered some problems when we tried to reduce the variance in the Monte Carlo simulation procedure. Because variance reduction in connection with Monte Carlo method is extremely important, there is a need for finding techniques which are effective in connection with time-dependent spot-rate models. The methods we tried here - which usually are very efficient - did not perform as promised/expected.

The only direct difference between the Hull and White lattice-based method and the Constrained Monte Carlo method introduced here - is how each of the procedures constructs the time-space dimension for the basic interest rate process x. From this it seems clear that the controlled method that is inherent in the lattice-based methodology gives a superior representation of the process than what is possible with Monte Carlo. The only logical explanation for this phenomonen must be that the spot-rate process is Markovian.

From this we conclude:

> Markovian spot-rate models are best represented by a method that takes into account the Markovian nature in the process - that is lattice-based methods Monte Carlo are not as efficient as lattice-based procedures for time-dependent Markovian spot-rate models, as Monte Carlo methods do not take into account the Markovian nature - Monte Carlo is doing precisely the opposite - as Monte Carlo methods by nature are non-Markovian.

I do not have any knowledge of any research that compares Monte Carlo pricing methods to lattice-based pricing methods for time-homogenous Markovian spot-rate models - but relying on the conclusion above it would not be a surprise if the same conclusions were reached[52].

In general it is worth pointing out that the pricing of path-dependent contingent claims does not rely on the assumption that the spot-rate process is non-Markovian - that is, the need for a non-Markovian pricing technique does not neccesarily indicate that the spot-rate process needs to be non-Markovian.

This indicates that when pricing path-dependent instruments we have to decide on the

[52] This could be an interesting line of research.

following two things:

1: How do we most efficiently construct the time-space dimension of the interest rate process?
2: Given that, how do we most efficiently utilize the constructed time-space dimension to price a path-dependent contingent claim?

For the pricing of path-dependent contingent claims there is a tradition for using Monte Carlo, because Monte Carlo methods by nature - like the problem at hand - is non-Markovian.

Doing that we do not distinguish between the two points mentioned above - but treat them as if they were one and the same. This is however not the case, as explained above. This does not neccesarily mean that Monte Carlo is not the most efficient method, it just indicates that we perhaps should re-think the problem - maybe there is a more efficient solution?.

10: Conclusion

Two of the main results in this paper were: First we showed how to determine the T-forward adjusted risk-measure using the concept of fundamental solution to linear PDEs. That is, it turned out to be possible to derive the T-forward adjusted risk-measure without the use of Girsanovs theorem, which is the traditional approach, see for example Karoui, Myneni and Viswanathan (1993).

The other main result was that we were able to carry the analysis of the Quadratic interest rate model further than Jamshidian (1996) by relying on the Fourier transformation in order to obtain the fundamental solution for the PDE.

In that connection we showed how it was possible to derive the price of a discount bond and the price of an option on a discount bond. This was also as an example done for the extended Vasicek model. Using the idea from Hull and White (1990), we showed how it was possible to fit the model to the initial term structure.

The next two parts of the paper focused on implemention issues.

In that connection, we designed a special discrete time model for the Quadratic interest rate model, as it in some cases was not possible to use the trinomial approach from Hull and White (1994).

After that we focused on pricing techniques for path-dependent interest rate contingent claims. We focused in that connection on Monte Carlo simulation of spot-rate models with a time-dependent drift. The third main result in the paper was to introduce a forward-induction technique that made it possible to constrain the Monte Carlo simulation for the matching of the initial yield-curve.

When using Monte Carlo simulation for time-dependent spot-rate models, we encountered problems obtaining variance reduction - by methods that in the case of time-homogenous spot-rate models are very effective. Further research is obviously needed in connection with Monte Carlo simulation of time-dependent spot-rate models.

Finally we compared lattice-based pricing method to Monte Carlo simulation procedures for the pricing of European put-and call-options on zero-coupon bonds for both the Extended Vasicek model and the Quadratic interest rate model.

In that connection we concluded:

> Markovian spot-rate models are best represented by a method that takes into account the Markovian nature in the process - that is lattice-based methods Monte Carlo are not as efficient as lattice-based procedures for time-dependent Markovian spot-rate models, as Monte Carlo methods do not take into account the Markovian nature - Monte Carlo is doing precisely the opposite - as Monte Carlo methods by nature are non-Markovian.

With respect to the pricing of path-dependent claims, no clear conclusion was drawn. It was however suggested that before deciding on a method (Monte Carlo or lattice-based method), it was of importance to separate the problem, as follows:

> How do we most efficiently construct the time-space dimension of the interest rate process?.
> Given that, how do we most efficiently utilize the constructed time-space dimension to price a path-dependent contingent claim?

From this we deduce that, even though Monte Carlo is the natural method to use when pricing path-dependent interest rate contingent claims, it might not be the most efficient one - at least not when the spot-rate is Markovian.

References:

Acworth, Broadie and Glasserman (1997) "A Comparison of some Monte Carlo and Quasi Monte Carlo Techniques for Option Pricing", in Niederreiter et al. (Eds.), Monte Carlo and Quasi-Monte Carlo Methods 1996 - Springer-Verlag New York, Lectures Notes in Statistics, 1998, pp.1-18

Ait-Sahalia (1995a) "Testing Continuous-Time Models of the Spot Interest Rate", Review of Financial Studies, 1996, 9, 385-426 (this paper received the Michael J. Brennan Award for the best paper published in the Review of Financial Studies in 1996)

Ait-Sahalia (1995b) "Nonparametric Pricing of Interest Rate Derivative Securities", Econometrica, 1996, 64, 527-560

Andersen (1995) "Efficient Techniques for Simulation of Interest Rate Models involving Non-Linear Stochastic Differential Equations", working paper Generel-Re-Financial Products

Andersen and Brotherton-Ratcliffe (1998) "The equity option volatility smile: an implicit finite-difference approach", The Journal of Computational Finance, vol. 1, no. 2, page 5-37

Andersen and Piterbarg (2010) "Interest Rate Modeling", Volume II, Atlantic Financial Press

Assefa (2007) "Calibration and pricing in a multi-factor Quadratic Gaussian Model", Research Paper 197, UTS, ISSN 1441-8010

Barraquand (1994) "Numerical Valuation of High Dimensional Multivariate European Securities", Management Science Vol. 41, No. 12, Dec., 1995

Beaglehole and Tenney (1991) "General Solution of some Interest Rate-Contingent Claim Pricing Equations", Journal of Fixed Income 1, page 69-83

Beaglehole and Tenney (1992) "A Nonlinear Equilibrium Model of the Term Structure of Interest Rates: Corrections and Additions", Journal of Financial Economics 32 (3), page 345-354

Black, Derman and Toy (1990) "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options", Financial Analyst Journal, January-February 1990, page 33-39

Black and Karasinski (1991) "Bond and Option pricing when short rates are lognormal", Financial Analyst Journal, July-August 1991, page 52-69

Black and Scholes (1973) "The Pricing of Options and Corporate Liabilities", Journal of Political Economy 81, page 637-654

Bloch and Assefa (2009) “Fast Calibration of interest rate claims in the quadratic gaussian model: 1 The Caplets”, working paper Mizuho Securities

Bloch and Assefa (2009) “Fast Calibration of interest rate claims in the quadratic gaussian model: 2 The Swaptions”, working paper Mizuho Securities

Bollerslev, Engle and Nelson (1993) “ARCH-Models”, working paper no. 154 - prepared for The Handbook of Econometrics, vol. 4

Boyle, Broadie and Glasserman (1995) “Monte Carlo Methods for Security Pricing”, Journal of Economic Dynamics and Control Volume 21, Issues 8-9, 29 June 1997, Pages 1267-1321

Brace, Gatarek and Musiela (1995) "The Market Model of Interest Rate Dynamics", Mathematical Finance, Vol. 7, No. 2 (April 1997), 127–147

Caflisch, Morokoff and Owen (1997) “Valuation of mortgage-backed securities using Brownian bridges to reduce effective dimensions”, The Journal of Computational Finance, vol. 1, no. 1, page 27-46

Campbell, Lo and Mackinlay (1997) “The Econometric of Financial Markets ”, Princeton University Press

Chen and Vincent Poor (2006) “Markovian Quadratic Term Structure Models for Risk-Free and Defaultable rates”, Working paper

Constantinides (1992) “A Theory of the Nominal Term Structure of Interest Rates”, Review of Financial Studies 5 (4), page 531-552

Cox and Miller (1965) “The Theory of Stochastic Processes”, Chapman and Hall. London

Duan and Simonato (1995) “Empirical Martingale Simulation for Asset Prices”, Management Science Vol. 44, No. 9, September 1998

Dupire (1994) “Pricing with a smile”, RISK 7, no. 1, page 18-20

Duffie (1996) "Dynamic Asses Pricing Theory", second edition, Princeton University Press

Flesaker and Houghston (1996) "Positive Interest: Foreign Exchange", working paper Merril Lynch, 10 of January 1996.

Gaspar (2004) “General Quadratic Term Structures of Bond, Futures and Forward Prices”, SSE/EFI Working paper series in Economics and Finance No 550, Stockholm School of Economics

Giles (2007) “Improved multilevel Monte Carlo convergence using the Milstein scheme”,

page 343-358, in Monte Carlo and Quasi-Monte Carlo Methods 2006, Springer, 2007

Gut (1995) "An intermediate course in Probability", Springer-Verlag New York

Heath, Jarrow and Morton (1991) "Bond Pricing and the Term Structure of Interest Rates: A new Methodology for Contingent Claim Valuation", Econometrica, Vol. 60, No. 1, (January 1992), pp 77-105.

Heitmann and Trautmann (1995) "Gaussian Multi-factor Interest Rate Models: Theory, Estimation, and Implications for Option Pricing", working paper Johannes Gutenberg-University Mainz June 1995, http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.40.7649

Hull and White (1990) "Pricing Interest-Rate-Derivative Securities", The Review of Financial Studies 3, page 573-592

Hull and White (1993a) "One-Factor Interest Rate Model and the Valuation of Interest Rate Derivative Securities", Journal of Financial and Quantitative Analysis 28, 1993, page 235-254

Hull and White (1993b) "Efficient Procedures for Valuing European and American Path-Dependent Options", Journal of Derivatives vol. 1, no.1 1993, page 21-31

Hull and White (1993c) “Bond Option pricing based on a model for the evolution of bond prices”, Advanced in Futures and Options Research, Vol. 6, page 1-13

Hull and White (1994) "Numerical Procedures for Implementing Term Structure Models", The Journal of Derivatives Fall 1994, Vol. 2, No. 1: pp. 7-16

Jackwerth and Rubenstein (1996) “Recovering Probability Distributions from Option Prices”, The Journal of Finance, vol. 51, no. 5, page 1611-1631

Jamshidian (1989) "An exact Bond Option Formula", Journal of Finance 44, page 205-209

Jamshidian (1995) "A Simple Class of Square-Root Interest Rate Models", Applied Mathematical Finance, vol. 2, no. 1, March 1995, page 61-72

Jamshidian (1996) "Bond, future and option evaluation in the quadratic interest rate model", Applied Mathematical Finance, vol. 3, no. 2, June 1996, page 93-116.

Jiang and Yan (2009) “Linear-Quadratic term structure models - toward the understanding of jumps in interest rates”, Journal of Banking and Finance 33, page 473-485

Karatzas and Schreve (1988) “Brownian Motion and Stochastic Calculus“, Springer-Verlag NewYork Inc.

Karoui, Myneni and Viswanathan (1993) "Arbitrage pricing and hedging of interest rate claims with state variables", Working Paper, University of Paris IV

Kijima and Nagayama (1995) "Efficient Numerical Procedures for the Hull-White Extended Vasicek Model", The Journal of Financial Engineering, vol. 3, no. 3/4, 1995, page 275-292

Kloeden and Platen (1995) “Numerical Solution of Stochastic Differential Equations”, Springer-Verlag Berlin Heidelberg, second corrected edition

Laurance and Wang (2005) “Closed form solutions for Quadratic and Inverse Quadratic Term Structure Models”, working paper

Leblon and Moraaux (2009) “Quadratic Term Structure Models: Analysis and Performance”, Working paper Université de Rennes

Leippold and Wu (1998) “Quadratic Term Structure Models”, Working paper, university of St. Gallen and University of New York

Leippold and Wu (2001) “Design and Estimation of Quadratic Term Structure Models”, CRIF Working paper series, 6-14-2001

Leippold and Wu (2002) “Asset Pricing under the Quadratic Class”, Journal of Financial and Quantitative Analysis, Vol. 37, No. 2, June 2002.

Longstaff (1989) “A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates”, Journal of Financial Economics 23, page 195-224

McDonnell (1998) “At Play with J: A Newer Random Link Generator”, Vector vol. 14, no. 4, April 1998

Matsumoto, M.; Nishimura, T. (1998). "Mersenne twister: a 623-dimensionally equidistributed uniform pseudo-random number generator". ACM Transactions on Modeling and Computer Simulation 8 (1): 3–30

Miltersen, Sandmann and Sondermann (1995) "Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates", The Journal of Finance, Vol. 52, No. 1, March 1997, pp. 409-430

Musiela and Rutkowski (1997) “Martingale Methods in Financial Modelling”, Springer-Verlag Berlin Heidelberg

Niederrieter (1992) “Random Number Generation and Quasi-Monte Carlo Method”, CBMS-NSF Regional Conference Series in Applied Mathematics 63

Paskov (1996) "New Methodologies for Valuing Derivatives", http://citeseer.ist.psu.edu/viewdoc/summary?doi=10.1.1.30.3094

Pelsser (1995) "An analytically tracable interest rate model that precludes negative interest rates", Springer Verlag, 2000

Pipkin (1991) "A Course on Integral Equations", Texts in Applied Mathematics 9, Springer-Verlag New York

Press, Teukolovski, Vetterling and Flannery (1992) “Numerical Recipes in C”, Cambridge University Press, second edition

Realdon (2006) “Quadratic Term Structure Models in Discrete Time”, working paper Department of Economics, University of York

Rosenberg (1997) "Pricing Multivariate Contingent Claims using Estimated Risk-Neutral Density Functions", http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1297773

Schoenmakers and Heemink (1997) “Fast valuation of financial derivatives”, The Journal of Computational Finance, vol. 1, no 1, page 47-62

Scott (1992) “Multivariate Density Estimation”, John Wiley and Sons, Inc.

Wilmott, Dewynne and Howison (1993) "Option Pricing - Mathematical models and computation", Oxford University Press

Zhenyu (1994) “Monte Carlo Simulation of the HJM Interest Rate Model”, working paper Cornell University

Appendix A

We seek a solution A(t,T,ς) and B(t,T,ς) that simultanously satisfies the following set of ordinary differential equation :

$$\begin{aligned} B_t - Ax(t) + \frac{1}{2}\sigma^2(t)A^2 - \psi(t) &= 0 \\ A_t - A\kappa(t) - 1 &= 0 \end{aligned} \qquad \textbf{(90)}$$

with respect to the boundary conditions A(T,T,ς) = iς and B(T,T,ς) = 0.

We will start with the second equation in formula 1 as we can here find a unique solution for A(t,T,ς). It is easily seen that A(t,T,ς) is of the following form:

$$\begin{aligned} A(t,T,\varsigma) &= -\int_t^T e^{k(s)}ds + k \\ &\text{where} \\ k(t) &= \int_t^T \kappa(s)ds \end{aligned} \qquad \textbf{(91)}$$

where k is the constant which can be found by using the boundary condition, hence we get:

$$\begin{aligned} A(t,T,\varsigma) &= -\int_t^T D(s,T)ds + i\varsigma D(t,T) \\ &\text{where} \\ D(t,T) &= e^{-\int_t^T \kappa(s)ds} \end{aligned} \qquad \textbf{(92)}$$

By plucking formula 3 into the first equation in formula 1 and rearranging terms we get:

$$B(t,T,\varsigma) = i\varsigma\int_t^T D(s,T)x(s)ds - \int_t^T A(s,T)x(s)ds$$
$$- \frac{1}{2}\int_t^T \sigma^2(s)[A(s,T)^2 - \varsigma^2 D(s,T)^2 - 2i\varsigma A(s,T)D(s,T)]ds + \int_t^T \psi(s)ds + k \quad \textbf{(93)}$$

Where k is the constant. Applying the boundary condition it can be seen that B(t,T,ς) is given by (as k = 0):

$$B(t,T,\varsigma) = \int_t^T \left[\frac{1}{2}\sigma^2(s)A(s,T)^2 + A(s,T)x(s) - \psi(sa)\right]ds$$
$$- \int_t^T \left[i\varsigma D(s,T)[x(s) + \sigma^2(s)A(s,T)] + \frac{1}{2}\varsigma^2\sigma^2(s)D(s,T)^2\right]ds \quad \textbf{(94)}$$

Formula 23 in the main text now follows directly from equation 3 and 5.

Appendix B

We seek a solution A(t,T,ς), B(t,T,ς) and C(t,T,ς) that simultanously satisfies the following set of ordinary differential equation :

$$\begin{aligned} A_t + \frac{1}{2}\sigma^2 B^2 - \sigma^2 C - \psi(t)^2 &= 0 \\ B_t - \kappa B - 2\sigma^2 BC + 2\psi(t) &= 0 \\ C_t - 2\kappa C - 2\sigma^2 C^2 + 1 &= 0 \end{aligned} \qquad \textbf{(95)}$$

with respect to the boundary conditions B(T,T,ς) = -iς and A(T,T,ς) = C(T,T,ς) = 0.

We will start with equation no. 3 in formula 1 as this is only a function of C(t,T,ς). C(t,T,ς) is known to be the solution to the following integral:

$$2(T - t) = \int_t^T \frac{1}{-\sigma^2 C(s,T,\varsigma)^2 - \kappa C(s,T,\varsigma) + \frac{1}{2}} dC \qquad \textbf{(96)}$$

which by employing a few tricks yields:

$$C(t,T,\varsigma) = C(t,T) = \frac{e^{2\gamma(T-t)} - 1}{(\gamma+\kappa)e^{2\gamma(T-t)} + (\gamma-\kappa)} \tag{97}$$

We will now turn our attention to the second equation in formula 1 as the only unknown here is B(t,T,ς), more precisely we are seeking a solution to the following first order ODE:

$$B_t = \kappa B + 2\sigma^2 BC(t,T) - 2\psi(t) \tag{98}$$

where it follows that B is given by:

$$\begin{aligned} B(t,T,\varsigma) &= -e^{-G(t,T)}\int_t^T 2\psi(s)e^{G(s,T)}ds + ke^{-G(t,T)} \\ &\text{where} \\ G(t,T) &= \int_t^T [2\sigma^2 C(s,T) + \kappa]ds \end{aligned} \tag{99}$$

The only problem here is the integral $\int_t^T C(s,T)ds$. In order to faciliate a solution let me first rewrite C(t,T) as:

$$\begin{aligned} C(t,T) &= \frac{\gamma-\kappa}{2\sigma^2}\left(\frac{(\kappa+\gamma)(e^{2\gamma(T-t)}-1)}{h(t,T)}\right) \\ &= \frac{1}{2\sigma^2}\frac{h'(t,T)}{h(t,T)} - \frac{\kappa+\gamma}{2\sigma^2} \\ &\text{where} \\ h(t,T) &= (\kappa+\gamma)e^{2\gamma(T-t)} + (\gamma-\kappa) \\ h'(t,T) &= 2\gamma(\kappa+\gamma)e^{2\gamma(T-t)} \end{aligned} \tag{100}$$

We can now easily derive $\int_t^T C(s,T)ds$ using the substitutions-rule which yields:

$$\int_t^T C(s,T)ds = \frac{1}{2\sigma^2}\left[\ln\left(\frac{2\gamma}{h(t,T)}\right) \mp (\kappa + \gamma)(T - t)\right] \quad \textbf{(101)}$$

By plucking formula 7 into formula 5 and doing a little algebra, we find that B(t,T,ς) can be written as:

$$B(t,T,\varsigma) = -2\int_t^T \frac{e^{\gamma s}h(s,T)}{e^{\gamma t}h(t,T)}\psi(s)ds + k\frac{2\gamma e^{\gamma(T - t)}}{h(t,T)} \quad \textbf{(102)}$$

The constant k can be found by employing the boundary condition B(T,T,ς) = -iς, hence we find the following expression for B(t,T,ς):

$$\begin{aligned} B(t,T,\varsigma) &= B(t,T) - i\varsigma D(t,T) \\ &\textit{where} \\ B(t,T) &= 2\int_t^T \frac{e^{\gamma s}h(s,T)}{e^{\gamma t}h(t,T)}\psi(s)ds \\ D(t,T) &= \frac{2\gamma e^{\gamma(T - t)}}{h(t,T)} \end{aligned} \quad \textbf{(103)}$$

Now we only have to determine A(t,T,ς) to have a solution to the system of ODE´s from formula 1, and it turns out that A(t,T,ς) with respect to the boundary condition can be written as:

$$A(t,T,\varsigma) = \int_t^T\left[\frac{1}{2}\sigma^2 B(s,T)^2 - \sigma^2 C(s,T) - \psi(s)\right]ds - \frac{1}{2}\sigma\int_t^T [2i\varsigma D(s,T)B(s,T) + \varsigma^2 D(s,T)^2]ds \quad \textbf{(104)}$$

Formula 39 and 40 in the main text now follows directly from equation 3,9 and 10.

Appendix C

From formula 45 in the main text, we have that the price at time t of a call-option with expiry at time T^F written on a discount bond that matures at time T, for $t < T^F < T$, in the Quadratic interest rate model is given by:

$$C(t,T^F) = P(t,T^F)\int_{-\infty}^{\infty}\left[\left[e^{A(T^F,T) - B(T^F,T)z - C(T^F,T)z^2} - K\right]^{+}\frac{1}{\sqrt{2\pi V(t,T^F)}}e^{-\frac{1}{2}\frac{[z - M(t,T^F)]^2}{V(t,T^F)}}\right]dz \quad \textbf{(105)}$$

This can be rewritten as:

$$C(t,T^F) = P(t,T^F)\int_{-\infty}^{\infty} e^{A(T^F,T) - B(T^F,T)z - C(T^F,T)z^2} p(q,t,T^F,z)dz - P(t,T^F)K\int_{-\infty}^{\infty} p(q,t,T^F,z)dz$$

$$where \qquad (106)$$

$$p(q,t,T^F,z) = \frac{1}{\sqrt{2\pi V(t,T^F)}} e^{-\frac{1}{2}\frac{[z - M(t,T^F)]^2}{V(t,T^F)}}$$

If we integrate formula 2 for all z for which the payoff is positive, we will be able to express the price of the call-option in terms of the cumulative normal distribution. From the main text we know that z has to be found as the root in a second order equation, more precisely we find that z is positive for values of z lying in the following interval:

$$Lower = \frac{-B(T^F,T) - \sqrt{Dis}}{2C(T^F,T)} < z < \frac{-B(T^F,T) + \sqrt{Dis}}{2C(T^F,T)} = Upper \qquad (107)$$

$$where$$

$$Dis = B(T^F,T)^2 + 4C(T^F,T)[A(T^F,T) - \ln K]$$

From this relation it it clearly seen that for Dis ≤ 0, the payoff H(z) will never be positive - that is the value of the option has to be equal to zero (0). If instead Dis > 0 we can integrate formula 2 over the region Lower < z < Upper.

The last integral in formula 2 is easy found as this can be expressed as:

$$\int_{Lower}^{Upper} p(q,t,T^F,z)ds = N\left(\frac{Upper - M(t,T^F)}{\sqrt{V(t,T^F)}}\right) - N\left(\frac{Lower - M(t,T^F)}{\sqrt{V(t,T^F)}}\right) \qquad (108)$$

From the first integral in formula 2 it follows that z has to multiplied by $1 + 2C(T^F,T)V(t,T^F)$. Which means that we need to multiply the variance and the b with $F = 1 + 2C(T^F,T)V(t,T^F)$, hence the first integral can be rewritten as[53]:

[53] Se for example Gut (1995).

$$\int_{-\infty}^{\infty} e^{A(T^F,T) - B(T^F,T)z - C(T^F,T)z^2} p(q,t,T^F,z)dz$$

$$= \left[\sqrt{F} e^{A(T^F,T) + \frac{\frac{1}{2}B(T^F,T)^2 V(t,T^F) - M(t,T^F)^2 C(T^F,T) - M(t,T^F)B(T^F,T)}{F}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V(t,T^F)F}} e^{-\frac{1}{2}\frac{[zF - M(t,T^F) + B(T^F,T)V(t,T^F)]^2}{V(t,T^F)F}} dz \right] \quad (109)$$

$$= \left[\sqrt{F} e^{A(T^F,T) + \frac{\frac{1}{2}B(T^F,T)^2 V(t,T^F) - B(T^F,T)M(t,T^F) - C(T^F,T)M(t,T^F)^2}{F}} \int_{\frac{LowerF - M(t,T^F) + B(T^F,T)V(t,T^F)}{\sqrt{FV(t,T^F)}}}^{\frac{UpperF - M(t,T^F) + B(T^F,T)V(t,T^F)}{\sqrt{FV(t,T^F)}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right]$$

As $P(T^F,T) = \sqrt{F} e^{A(T^F,T) + \frac{\frac{1}{2}B(T^F,T)^2 V(t,T^F) - B(T^F,T)M(t,T^F) - C(T^F,T)M(t,T^F)^2}{F}}$

expression:

$$P(T^F,T)\left[N\left(\frac{UpperF - E}{\sqrt{FV(t,T^F)}} \right) - N\left(\frac{LowerF - E}{\sqrt{FV(t,T^F)}} \right) \right] \quad (110)$$

All in all this means that the call-options price is given by:

$$C(t,T^F) = P(t,T)\left[N\left(\frac{UpperF - E}{\sqrt{V(t,T^F)F}} \right) - N\left(\frac{LowerF - E}{\sqrt{V(t,T^F)F}} \right) \right]$$
$$- P(t,T^F)K\left[N\left(\frac{Upper - M(t,T^F)}{\sqrt{V(t,T^F)}} \right) - N\left(\frac{Lower - M(t,T^F)}{\sqrt{V(t,T^F)}} \right) \right] \quad (111)$$

where

$$E = M(t,T^F) - B(T^F,T)V(t,T^F)$$
$$F = 1 + 2C(T^F,T)V(t,T^F)$$

Using a similar technique it is possible to derive the price of a put-option, hence we get:

$$Put(t,T^F) = P(t,T^F)K\left[1 - \left[N\left(\frac{Upper - M(t,T^F)}{\sqrt{V(t,T^F)}} \right) - N\left(\frac{Lower - M(t,T^F)}{\sqrt{V(t,T^F)}} \right) \right] \right]$$
$$- P(t,T)\left[1 - \left[N\left(\frac{UpperF - E}{\sqrt{V(t,T^F)F}} \right) - N\left(\frac{LowerF - E}{\sqrt{V(t,T^F)F}} \right) \right] \right] \quad (112)$$

where

$$E = M(t,T^F) - B(T^F,T)V(t,T^F)$$
$$F = 1 + 2C(T^F,T)V(t,T^F)$$

Appendix D

We are interested in a solution $\psi(T)$ that satisfies the following integral equation:

$$\psi(T) - K(s,v) = F(T)$$

where

$$K(s,v) = 2\sigma^2 \int_0^T \frac{2\gamma e^{\gamma(T-s)}}{h(s,T)} \int_v^T \frac{e^{\gamma v} h(v,T)}{e^{\gamma s} h(s,T)} \psi(s)\,dv\,ds \qquad \textbf{(113)}$$

$$F(T) = \sqrt{r\,F_{(0,T)} - \frac{1}{2}V(0,T) + \frac{1}{2}V(0,T)}$$

If it is possible to write K(s,v) as $\int_0^T K(T,v)\psi(v)dv$ then equation no. 1 can be recognized as a linear second order Volterra integral with a seperable kernel[54]. In that connection, it is known that a linear second order Volterra integral with a continous and bounded kernel has a solution for every continous function F(T).

We are therefore seeking a solution of the following form:

$$\psi(T) - \int_0^T K(T,v)\psi(v)dv = F(T) \qquad \textbf{(114)}$$

Let us then take a more close look at the following integral equation:

$$2\sigma^2 \int_0^T \frac{2\gamma e^{\gamma(T-s)}}{h(s,T)} \int_v^T \frac{e^{\gamma v} h(v,T)}{e^{\gamma s} h(s,T)} dv\,ds \qquad \textbf{(115)}$$

where

$$h(t,T) = [\kappa + \gamma]e^{2\gamma(T-t)} + [\gamma - \kappa]$$

This formula can (after lengthly calculation) be shown to be possible to rewrite as follows:

$$2\sigma^2 \int_0^T \frac{2\gamma e^{\gamma(T-s)}}{h(s,T)} \int_v^T \frac{e^{\gamma v} h(v,T)}{e^{\gamma s} h(s,T)} dv\,ds = \frac{2\sigma^2 e^{\gamma T}}{h(0,T)} \int_0^T [e^{\gamma v} - e^{-\gamma v}]dv \qquad \textbf{(116)}$$

[54] See Pipkin (1991) chapter 5.

We are then in a situation were we have to find a solution ψ(T) that satisfies the following linear second order Volterra integral:

$$\psi(T) - \int_0^T K(T,v)\psi(v)dv = F(T)$$
$$where \qquad (117)$$
$$K(T,v) = \frac{2\sigma^2 e^{\gamma T}}{h(0,T)}[e^{\gamma v} - e^{-\gamma v}]$$

In the case of a seperable kernel (as here) the integral equation can be solved as follows. First we differentiate formula 5 with respect to T, which yields:

$$\psi_T(T) - K(T,T)\psi(T) - 2\sigma^2 \frac{\gamma e^{\gamma T} h(0,T) - e^{\gamma T} h_T(0,T)}{h(0,T)^2} \int_0^T [e^{\gamma v} - e^{-\gamma v}]\psi(v)dv = F_T(T) \qquad (118)$$
$$where$$
$$h_T(0,T) = 2\gamma[\kappa + \gamma]e^{2\gamma T}$$

If we then use formula 5, we can express $\int_0^T [e^{\gamma v} - e^{-\gamma v}]\psi(v)dv$ as:

$$\int_0^T [e^{\gamma v} - e^{-\gamma v}]\psi(v)dv = [\psi(T) - F(T)]\frac{h(0,T)}{2\sigma^2 e^{\gamma T}} \qquad (119)$$

If we then pluck formula 7 into formula 6, we get:

$$\begin{aligned}
&\psi_T(T) - \frac{2\sigma^2 e^{\gamma T}}{h(0,T)}[e^{\gamma T} - e^{-\gamma T}]\psi(T) - [\psi(T) - F(T)]\frac{\gamma h(0,T) - h_T(0,T)}{h(0,T)} = F_T(T) \\
&\Rightarrow \psi_T(T) - \psi(T)\left[\frac{2\sigma^2 e^{\gamma T}}{h(0,T)}[e^{\gamma T} - e^{-\gamma T}] + \gamma - \frac{2\gamma[\kappa + \gamma]e^{2\gamma T}}{h(0,T)}\right] \\
&= F(T)\left[\gamma - \frac{2\gamma[\kappa + \gamma]e^{2\gamma T}}{h(0,T)}\right] + F_T(T) \\
&\Rightarrow \psi_T(T) + \psi(T)\kappa = F_T(T) - F(T)\left[\gamma - \frac{2\gamma[\kappa + \gamma]e^{2\gamma T}}{h(0,T)}\right] \\
&\Rightarrow \psi_T(T) + \kappa\psi(T) = F(T) + F_T(T)W(T) \\
&where \\
&W(T) = \gamma - \frac{2\gamma[\gamma - \kappa]}{h(0,T)}
\end{aligned} \tag{120}$$

That is ψ(T) can be found by solving an ordinary differential equation subject to the boundary condition that ψ(0) = F(0). The solution to this equation can with respect to that be written as:

$$\psi(T) = e^{-\kappa T}\left[F(0) + \int_0^T e^{\kappa s}W(s)F(s)ds + \int_0^T e^{\kappa s}F_S(s)ds\right] \tag{121}$$

The last integral in formula 9 can be rewritten as follows by using partial integration:

$$\int_0^T e^{\kappa s}F_S(s)ds = e^{\kappa T}F(T) - F(0) - \kappa\int_0^T e^{\kappa s}F(s)ds \tag{122}$$

Which means that:

$$\psi(T) = F(T) + e^{-\kappa T}\int_0^T e^{\kappa s}[W(s) - \kappa]F(s)ds \tag{123}$$

If we pluck the expression for W(T) from formula 8 into this formula and simplify we finally find that ψ(T) for all T is given by:

$$\psi(T) = F(T) + 2e^{-\kappa T}\int_0^T e^{\kappa s} V(0,s)F(s)ds \qquad \textbf{(124)}$$

Which is identical to formula 55 in the main text.

Appendix E

From equation 51 in the main text we have that $r^{F}(t,T) = E^{Q_T}[r(T)|F_t]$ and furth

we have from formula 24 that the state-variable (y) in the extended Vasicek model is

governed by the following stochastic process under the T-forward-adjusted probability measure:

$$y(T) = D(t,T)y_t - \sigma^2\int_t^T D(s,T)A(s,T)ds + \sigma\int_t^T D(s,T)d\tilde{W}^{Q_T} \quad \textbf{(125)}$$

As r(T) = y(T) + ψ(T) we have:

$$r(T) = D(t,T)[r_t - \psi(t)] + \psi(T) - \sigma^2\int_t^T D(s,T)A(s,T)ds + \sigma\int_t^T D(s,T)d\tilde{W}^{Q_T} \quad \textbf{(126)}$$

Now we want to express the stochastic process for the instantaneous forward rate under the Q_T-probability measure in terms of the value of the state-variable (including ψ(T)) at time T, that is:

$$r^F_{(t,T)} = r(T) + \sigma\int_t^T D(s,T)d\tilde{W}^{Q_T} \quad \textbf{(127)}$$

Combining formula 2 and formula 3 yields:

$$r^F_{(t,T)} = D(t,T)[r_t - \psi(t)] + \psi(T) - \sigma^2\int_t^T D(s,T)A(s,T)ds + \sigma\int_t^T D(s,T)d\tilde{W}^{Q_T} \quad \textbf{(128)}$$

For t = 0 we then find that ψ(T) is defined by:

(129)

Which under the assumption that κ(t) and σ(t) are time-independent yields:

$$\psi(T) = r^F_{(0,T)} + \frac{\sigma^2}{2\kappa^2}[1 - e^{-\kappa T}]^{T2} \quad \textbf{(130)}$$

If we want to find an expression for B(t,T) and A(t,T) in the extended Vasicek model it is most efficient to use the following relationship:

$$r^F(t,T) = -B_T(t,T) + A_T(t,T)r_t = D(t,T)[r_t - \psi(t)] + \psi(T) - \sigma^2\int_t^T D(s,T)A(s,T)ds \quad \textbf{(131)}$$

Which means that $B_T(t,T)$ and $A_T(t,T)$ can be found by collecting powers of r_t, which yields:

$$A_T(t,T) = D(t,T)$$

$$B_T(t,T) = D(t,T)r^F(0,t) + \sigma^2 D(t,T)\int_0^t D(s,t)A(s,tT)ds - r^F(0,T) - \sigma^2\int_0^t D(s,T)A(s,T)ds \quad \textbf{(132)}$$

If this expression is integrated from 0 - T (under the assumption of constant $\kappa(t)$ and $\sigma(t)$) and plucked into formula 25 in the main text the result will be identical the result from Hull and White (1993c).

Another way to derive the expression for B(t,T) and A(t,T) which actually is easier as it does not involve much integration because we are cabable of writing the definitions of B(t,T) and A(t,T) in terms of B(0,T), B(0,t) and B(0,T), B(0,t), the technique is as follows:

We have the following well known result:

$$P(t,T) = P(t,T_1)E^{Q_{T_1}}_{[P(T_1,T)|F_t]} \quad \textbf{(133)}$$

This can alternatively be written as:

$$e^{B(t,T) - B(t,T_1) - [A(t,T) - A(t,T_1)]y} = E^{Q_{T_1}}_{[P(T_1,T)|F_t]} \quad \textbf{(134)}$$

We have from formula 30 in the main text (for $T_1 = T^F$) that the price $P(T_1,T)$ under the probability measure Q_{T_1} can be written as:

$$e^{B(T_1,T) + \frac{12}{2}A(T_1,T)^2V(t,T_1) - A(T_1,T)M(t,T_1)} \quad \textbf{(135)}$$

Combining this equation with formula 10 and collecting terms of equal power in y, we get the following formulas for $B(T_1,T)$ and $A(T_1,T)$:

$$A(T_1,T) = \frac{A(t,T) - A(t,T_1)}{D(t,T_1)}$$
$$B(T_1,T) = [B(t,T) - B(t,T_1)] - \frac{1}{2}A(T_1,T)^2 V(t,T_1) - A(T_1,T)[\psi(T_1) - r^F(0,T_1)] \quad \textbf{(136)}$$

For t = 0 we have that B(0,T) = ln P(0,T) more precisely we get the following expressions for $B(T_1,T)$ and $A(T_1,T)$:

$$A(T_1,T) = \frac{A(0,T) - A(0,T_1)}{D(0t,T_1)}$$
$$B(T_1,T) = \ln\left(\frac{P(0,T)}{P(0,T_1)}\right) - \frac{1}{2}A(T_1,T)^2 V(0t,T_1) - A(T_1,T)[\psi(T_1) - r^F(0,T_1)] \quad \textbf{(137)}$$

In order to have a fully defined model, we now only have to find the expression for A(0,T) for all T.

We can get the expression for A(0,T) by using formula 8 and find that A(0,T) is defined as:

$$A(0,T) = \int_0^T D(0,s)ds \quad \textbf{(138)}$$

It might not be obvious but formula 13 (together with formula 14) evaluated at $t = T_1$, and under the assumption that κ(t) and σ(t) are time-independent yields identical results for B(t,T) and A(t,T) as the expression from formula 8.

Appendix F

In terms of forward-rates we can in abstract form write the Quadratic Interest rate model as[55]:

$$r^F(t,T) = b_1(t,T)y_t^2 + b_2(t,T)y_t + a(t,T) \quad \textbf{(139)}$$

Where the process for y_t is defined by the following SDE[56]:

$$\begin{gathered} dy_t = [\varphi(t) - \kappa y_t]dt + \sigma d\tilde{W}_t \\ \textit{where} \\ d\tilde{W}_t = dW_t - \lambda\sigma \end{gathered} \quad \textbf{(140)}$$

The term structure appears as a quadratic function of the factor y_t - which follows a Vasicek type process. Let us in this connection remark that the unconditional correlation of y_t^2 and y_t is not 1 - as a consequence of this the Quadratic Interest rate model behaves - from an empirical point of view - like a linear two-factor model.

[55] The expression for $b_1(t,T)$, $b_2(t,T)$ and $a(t,T)$ can be derived from formula 37 in the main text.

[56] Se formula 32 in the main text.

Let us therefore consider the following two-factor linear model[57]:

$$r^F(t,T) = a_1(t,T)f_{1t} + a_2(t,T)f_{2t} + b(t,T) \quad \textbf{(141)}$$

From this it can be seen that the Quadratic Interest rate model can be thought of as a linear factor model, where $f_{1t} = f_{2t}^2$.

In general we can write the vector process for f_t as:

$$df_t = [\Phi(t) - Bf_t]dt + S(F_t)d\tilde{W}_t \quad \textbf{(142)}$$

In particular we have:

$$df_{2t} = [\varphi_2(t) - \kappa_1 f_{1t} - \kappa_2 f_{2t}]dt + \sigma_2(f_t)^T d\tilde{W}_t \quad \textbf{(143)}$$

And, since $f_{1t} = f_{2t}^2$ we have:

$$df_{1t} = [2f_{2t}[\varphi_2(t) - \kappa_1 f_{1t} - \kappa_2 f_{2t}] + \sigma_2(f_t)^2]dt + 2f_{2t}\sigma_2(f_t)^T d\tilde{W}_t \quad \textbf{(144)}$$

Equation 5 and 6 imply that:

$$S(f_t)S(f_t)^T = \sigma_2(f_t)^2 \begin{pmatrix} 4f_{2t}^2 & 2f_{2t} \\ 2f_{2t} & 1 \end{pmatrix} \quad \textbf{(145)}$$

From this we can deduce that the matrix $(S(f_t)S(f_t)^T)$ is linear in terms of f_t - if and only if $\sigma_2(f_t)^2$ does not depend on f_t. This is the explanation why the Quadratic Interest rate model

[57] Where the functional form for $b_1(t,T)$, $b_2(t,T)$ and $a(t,T)$ can be derived from the closed form solution for zero-coupon bond prices which is implied by the SDE for the two-factors f_{1t} and f_{2t} and their linear connection with the spot-rate, i.e, $r_t = x^T f_t$ - where x is a Boolean vector.

needs a deterministic volatility for the state-variable. Furthermore, the drift for f_{1t} should be linear in f_t, and then $\kappa_1 = 0$.

This means that the Quadratic Interest rate model is equivalent to a two-factor linear framework where there is only one state-variable f_{2t} - following a Vasicek process. That is, the term structure depends linearly on f_{2t} and f_{2t}^2 .

To conclude these results, let me remark that the Quadratic Interest rate model appears a particular kind of two-factor linear model, satisfying:

$$r^F(t,T) = b(t,T)^T f_t + a(t,T) \qquad \textbf{(146)}$$

where there exists a quadratic function g such that:

$$\begin{gathered} g(f_t) = 0 \\ for \\ g(f_t) = f_{2t}^2 - f_{1t} \end{gathered} \qquad \textbf{(147)}$$

That is the Quadratic Interest rate model can be thought of as a particular case of the class of linear factor models.

Appendix G

In this Appendix I will shortly present the different variance reduction methods that I have employed in some of the examples in section 9.

This Appendix will consist of two parts.

The first part will discuss the variance reduction methods I am using in simulating the forward-price in the Extended Vasicek model under the T^F-adjusted probability measure. The problem considered here can be recognized as a 1-dimensional simulation problem.

The second part will present the variance reduction methods which were used in connection with sampling the spot-rate process from a discretization of the SDE. This problem can for this reason considered a K-dimensional simulation problem, where K is the number of steps. As we only consider equidistant discretization points, we have that the time-step δ is equal to $\frac{T-t}{K}$, for T-t being the total sampling interval.

This does not neccesarily mean that the methods employed in part 1 cannot be used on the problem considered in part 2. Actually there is nothing to stop us from doing exactly that.

On the other hand the approaches I will discuss in part 2 are not useful for the particular type of 1-dimensional problem I am considering. This will be obvious from the presentation in part 2.

The variance reduction method I will discuss in part 1 is:

Antithetic variates
Stratified sampling
Empirical Martingale simulation

For the more complex simulation problem considered in part 2 I have chosen two new methods which looks promising:

The Brownian Bridge technique
Girsanovs transformation

Part 1: Variance Reduction Techniques - I[58]

The equation we are considering in this example is given by formula 86 in the main text, ie.:

[58] The discussion in this section relies mainly on Duan and Simonato (1998) and Boyle, Broadie amd Glasserman (1995).

$$P(T^F,T) = \frac{P(0,T)}{P(0,T^F)} \exp\left[-\frac{1}{2} \int_0^{T^F} [\sigma_P(s,T) - \sigma_P(s,T^F)]^2 ds - \int_0^{T^F} [\sigma_P(s,T) - \sigma_P(s,T^F)] dW^{Q_{TF}}(s) \right] \quad (148)$$

Where $dW^{Q_{TF}} \approx N(0,1)$. Let us in that connection denote z_n, for n = [1,N], the vector of rando numbers simulated from the distribution N(0,1).

Antithetic variates:

One of the simplest and most widely used variance reduction methods in finance is the method of antithetic variates.

The method of antithetic variates is based on the observation that if z_n, has a standard normal distribution, then so does $-z_n$.

Let us now assume that we have simulated the bond-price $P(T^F,T)$ for a given N - let us denote this simulated price as $P^+(T^F,T)$. If we now also simulate the price where we have replaced each z_n with $-z_n$ - and denote this price $P^-(T^F,T)$ - then we have that an unbiased estimator of the forward price is given by:

$$P(T^F,T) \approx \tilde{P}(T^F,T) = \frac{1}{N} \sum_{n=1}^{N} \frac{P^+(T^F,T) + P^-(T^F,T)}{2} \quad \textbf{(149)}$$

We have that the random inputs obtained from the collection of antithetic pairs are more regularly distributed than a collection of 2N independent samples. In particular, we have that the sample mean over the antithetic pairs will always equal the population mean of 0, whereas the mean over finitely many independent samples is almost surely different from 0.

More precisely we have the following relationship, because $P^+(T^F,T)$ and $P^-(T^F,T)$ has the same variance:

$$Var\left[\frac{P_n^+(T^F,T) + P_n^-(T^F,T)}{2} \right] = \frac{1}{2} [Var[P_n^+(T^F,T)] + Cov[P_n^+(T^F,T), P_n^-(T^F,T)]] \quad (150)$$

From formula 3 it follows that $Var[\tilde{P}(T^F,T)] \leq Var[P^+(T^F,T)]$ if $Cov[P_n^+(T^F,T),\ P_n^-(T^F,T)] \leq Var[P_n^+(T^F,T)]$.

Because we need twice as many replications to calculate/estimate $\tilde{P}(T^F,T)$ - that is twice the amount of work. We require the following inequality to hold in order for antithetic variates to have increased the efficiency, $2Var[\tilde{P}(T^F,T)] \leq Var[P^+(T^F,T)]$ - whic equation 3 simplifies to the following requirement; $Cov[P_n^+(T^F,T),\ P_n^-(T^F,T)] \leq 0$ the two simulated prices should be negatively correlated[59].

Stratified sampling:

The idea behind stratified sampling is as follows: stratified sampling seeks to make the inputs to the simulation more regular than random inputs. In particularly it forces certain empirical probabilities to match theoretical probabilities[60].

Let us consider N simulated normal random numbers. The empirical distribution of an independent sample (z_1, z_2, z_3,...., z_N) will look only roughly like the normal density function, the tails of the distribution - which often is the most important part - will inevitably be under represented.

Using stratified sampling we can force exactly one observation to lie between the $(n - 1)^{th}$ and n^{th} percentile, for n = [1,N]. This will - by construction - produce a better match to the normal distribution.

The way we have chosen to implement this method is as follows: First we generate N independent uniform random variates (u_1, u_2, u_3,...., u_N) - uniform on [0,1]. The vector of independent normal random variates can now be derived from the following formula:

$$z_n = N^{-1}\left(\frac{n + u_n - 1}{N}\right) \qquad for\ n\ \varepsilon\ [1,N] \qquad \textbf{(151)}$$

Where N^{-1} is the inverse of the cumulative normal distribution.

[59] A proof that this requirement is always met is given in Boyle, Broadie and Glasserman (1995).

[60] This is in a sense similar to the idea behind moment matchning methods - which force the empirical moments to match the theoretical moments, see Barraquand (1994).

The reason that this works is because $\frac{n + u_n - 1}{N}$ falls between the $(n - 1)^{th}$ and n^{th} percentile of the uniform distribution and percentiles are preserved by the inverse transformation.

Empirical Martingale simulation:

The basic idea behind the Empirical Martingale Simulation (EMS) method from Duan and Simonato (1998) can be explained this way:

Lets us for this purpose consider the exponential part of equation 1:

$$\tilde{A} = \frac{1}{N}\exp\left[-\frac{1}{2}\int_0^{T^F}[\sigma_P(s,T) - \sigma_P(s,T^F)]^2 ds - z_n\int_0^{T^F}[\sigma_P(s,T) - \sigma_P(s,T^F)]\right] \quad \textbf{(152)}$$

Under the Q_{TF}-probability measure we have that the expected value of $\tilde{A}$ is 1. Of course for a finite number of simulated normal random variates the average value calculated in formula 5 is almost surely different from 1.

The idea in the EMS-method is, because of this observation, to adjust each element in the vector $(a_1, a_2, a_3, ..., a_N)$, for $\tilde{A} = \frac{1}{N}\sum_{n=1}^{N} a_n$ - that is each a_n can be recognized as a particular realisation of the exponential part in equation 5. The adjustment is done to ensure that we for all N have that: $\tilde{A} = 1$.

From this it follows that EMS bears some resemblance to the moment matching method of Barraquand (1994). However, the EMS method is entirely different from the moment matching method because of the following reason: For the EMS the correct first moment in simulation is ensured by the use of a multiplicative adjustor instead of an additive one as in moment matching[61]. This difference is very important as asset prices typically (as here) are modelled as exponential martingales. From this it follows that the multiplicative adjustment ensures no domain violation, whereas the additive adjustment cannot.

[61] When moment matching is applied to asset prices.

Part 2: Variance Reduction Techniques - 2[62]

The equation we are considering here is given by formula 84 in the main text, ie:

$$dx_t = -\kappa x_t dt + \sigma dW_t \quad for\ x_0 = 0 \tag{153}$$

where $dW_t \approx \Delta W_k = W_{(k+1)\delta} - W_{k\delta}$ is distributed as N(0,δ). There

$\Delta W_k = z_k\sqrt{\delta}$.

As mentioned above we assume that K is the number of steps, and the time discretization points are of an equal length δ. Furthermore we denote each discretization point as kδ, for k = [1,K]. Lastly we assume that we for each k perform N simulations.

The Brownian Bridge technique:

As far as I know the only other work that uses the Brownian Bridge method to reduce the effective dimensionality is Caflisch, Morokoff and Owen (1997). Let me for that reason explain the idea behind using the Brownian Bridge method in some detail[63]:

For convenience I will assume that the number of time-steps is a power of 2. Let $Z(n) = (z_1(n), z_2(n), z_3(n),...., z_K(n))$, for n = [1,N], be a vector of independent normal random variates - that is Z(n) can be seen as a particular path for the SDE over the interval [δ,Kδ].

This vector we will use to construct a Brownian Bridge simulated Wiener process, which we will denote B_k, for k = [0,K]. In particular we have that $B_0 = 0$. We will in the construction assume that the quality of the simulated normal random variates drops with the dimension number[64].

As the variance of a Wiener process is linear in the time-length - the first point to generate will be B_K, ie:

$$B_K = z_K(n)\sqrt{K\delta} \tag{154}$$

[62] The discussion in this section relies mainly on Caflisch, Morokoff and Owen (1997) and Schoenmakers and Heemink (1997).

[63] The descriptions of the Brownian Bridge method rely on Caflisch, Morokoff and Owen (1997) and Karatzas and Shreve (1988, section 2.3).

[64] This is generally the case, at least in the case of quasi-random number sequences.

We have now got both the starting point and the end point for the Wiener process. Viewing these as fixed points, we can think of the Wiener process as being "tied-down" - this can be recognized as a Browning Bridge process.

The next point to simulate will be the one that has the highest variance, which for the Brownian Bridge process is the midpoint of the interval. Conditioned on B_0 and B_K, the midpoint is normally distributed as $N\left(\frac{1}{2}B_K, \frac{1}{4}K\delta\right)$, which mean that we can simulate the midpoint by setting:

$$B_{\frac{K}{2}} = \frac{B_K}{2} + z_{\frac{K}{2}}(n)\sqrt{\frac{K\delta}{4}} \qquad \textbf{(155)}$$

We now have two endpoints and a midpoint. This can in a sense be thought of as two consecutive Brownian Bridge processes - conditioned on each of the intervals.

Again we will simulate the process where the variance is highest, that is at the midpoints of each of the two intervals.

In general we have that each B_k can be derived from the following formula (if K is of the power of 2)[65]:

$$B_k = \frac{B_{(k-1)} + B_{(k+1)}}{2} + z_k(n)\sqrt{\frac{K\delta}{2^v}} \quad for\ v\ \varepsilon\left(1,\ 1 + \frac{\ln K}{\ln 2}\right) \qquad \textbf{(156)}$$

After having simulated our K+1-dimensional vektor B_k for each k = [0,K], we can construct our simulated Wiener process from this. Actually we have that we can form K-simulated Wiener increments as follows:

$$\Delta W_k = B_k - B_{k-1} \quad for\ k\ \varepsilon\ [1,K] \qquad \textbf{(157)}$$

Accordingly to Corollary 2.3.4 in Karatzas and Shreve (1988) this should produce paths that are asymptotical Wiener paths.

[65] See also equation 5.2 in Caflisch, Morokoff and Owen (1997).

This new constructed vector of Wiener increments can now directly be used instead of $\Delta W_k = z_k\sqrt{\delta}$ in our simulation procedure.

Variance reduction using Girsanovs transformation:

I will now turn my attention to the Girsanovs transformation technique as a means of reducing the variance in the simulation procedure.

This method is based on a Girsanov transformation of the original SDE for x.

Equation 6 can alternatively be written as:

$$x_t = x_0 - \int_0^t \kappa x_s ds + \int_0^t \sigma dW_s \quad for\ x_0 = 0 \tag{158}$$

The basic idea in the weak discretizations scheme is to approximate the functional:

$$u(s,x) = E[h(x_T)|x_s = x] \tag{159}$$

where h is a specific function and time s = 0. If we have that the function h and the drift-and diffusion-coefficients in formula 11 are sufficiently smooth, then the function u satisfies the Kolmogorov backward equation, ie:

$$L^0 u(s,x) = 0 \tag{160}$$

Subject to the end condition $u(T,x) = h(x)$ for all x ε ℜ, and where the operator L^0 is defined as:

$$L^0 = \frac{\partial}{\partial s} + \mu\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} \tag{161}$$

$$where$$
$$\mu = -kx$$

Using Girsanovs theorem we can transform the underlying probability measure P, so that the process $\tilde{W}$ is defined by:

$$\tilde{W} = W - \int_0^t v(s,x_s)ds \qquad \textbf{(162)}$$

is a Brownian motion with respect to the transformed probability measure $\tilde{P}$ with Radon-Nikodym derivative:

$$\frac{d\tilde{P}}{dP} = \frac{\Theta_t}{\Theta_0} \qquad \textbf{(163)}$$

Under the change of probability measure the Ito process $\tilde{x}$ satisfies the stochastic differential equation:

$$\tilde{x}_t = \tilde{x}_0 - \int_0^t [\kappa\tilde{x}_s + v(s,\tilde{x}_s)\sigma]ds + \int_0^t \sigma dW_s \quad for\ \tilde{x}_0 = 0 \qquad \textbf{(164)}$$

Furthermore we have that the process Θ satisfies the equation:

$$\Theta_t = \Theta_0 + \int_0^t \Theta_s v(s,\tilde{x}_s)dW_s \quad Where\ \Theta_0 \neq 0 \qquad \textbf{(165)}$$

It is obvious that the process $\tilde{x}$ in equation 17 is an Ito process with respect to $\tilde{P}$ with the same drift-and diffusion coefficients as the Ito process x in formula 11. From this and equation 16 it follows that:

$$u(s,x) = E[h(x_T)] = \frac{E[h(\tilde{x}_T)]\Theta_T}{\Theta_0} \qquad \textbf{(166)}$$

This indicates that by estimating the expectation of the random variable om the right hand side of equation 19 we are evaulating the functional given in 12.

Furthermore this result is independent of the function v. The question is therefore how to choose v to get the most efficient variance reduction.

Given the Radon-Nikodym derivative, we can by using Ito's lemma deduce that the product $u(s,\tilde{x})\Theta_t$ under the probability measure P, follows:

$$d(u(s,\tilde{x})\Theta_t) = \Theta_t\left(u(s,\tilde{x})v(s,\tilde{x}) + u_x(s,\tilde{x})\sigma\right)dW_s \quad \textbf{(167)}$$

What we hope for is that the variance of the right side of equation 19 is much smaller than the variance of u(s,x). It follows immediately from formula 20 that this variance is reduced to zero if $v(s,\tilde{x})$ satisfies:

$$v(s,\tilde{x}) = -\frac{u_x(s,\tilde{x})\sigma}{u(s,\tilde{x})} \quad \textbf{(168)}$$

In which case we have[66]:

$$u(0,x) = \frac{E[h(\tilde{x}_T)]\Theta_T}{\Theta_0} \quad \textbf{(169)}$$

That is the variable is non-random and therefore the variance is reduced to zero.

For the construction of the parameter function in 21 we need to know the solution u of the Kolmorogov backward equation.

In our particular case it happens to be possible to obtain the solution - the procedure is as follows:

Let us assume that x represents the spot-rate and we want to derive the price of a zero-coupon bond - $P_b(t,T)$, for all $T = [1,\tau]$ where τ is the maximum maturity date. The end condition for the Kolmogorov backward equation[67] will in that case be $P_b(T,T) = u(T,x) = h(x) = 1$

From Duffie (1992 Appendix E) we have that the Feynman-Kac solution to the Kolmogorov backward equation under the above assumption - if it exists - can be written as:

[66] The proof is here obmitted, but can be derived from the SDE's from equation 17 and 18, together with 21 and 13, see Kloeden and Platen (1995 section 16.2).

[67] It should here be stressed that the Kolmogorov backward equation in the case of the pricing of zero-coupon bonds given a spot-rate process is not identical to the one specified in formula 14. For our purpose - namely figuring out the functional from for v - this is not of importance, as will be apparent in the derivation. It would be important if we tried to solve the PDE directly - but here we are utilising a probabilistic approach.

$$P_b(t,T) = u(t,x) = E\left[\exp -\int_t^T x_s ds\right] \tag{170}$$

As x is normally distributed, the solution to 23 can be written as:

$$P_b(t,T) = u(t,x) = e^{-E[\int_t^T x_s ds] + \frac{1}{2}V[\int_t^T x_s ds]} \tag{171}$$

We know that the expected value and variance for $Y(t,T) = \int_t^T x_s ds$, can be written as:

$$\begin{aligned} E[Y(t,T)] &= \int_t^T e^{-\kappa(v-t)}\left[x + \int_t^v e^{\kappa(s-t)}\mu ds\right]dv \\ V[Y(t,T)] &= \int_t^T e^{-2\kappa(v-t)}\left(\int_t^v e^{\kappa(s-t)}\sigma ds\right)^2 dv \end{aligned} \tag{172}$$

where we on purpose have specified the drift in an abstract form.

From this we can easily derive the functional form for v as $\frac{u_x(s,\tilde{x})}{u(s,\tilde{x})} = \frac{u_x(s,x)}{u(s,x)}$ - it follows directly from formula 21, 24 and 25 - and the result is:

$$v(s,\tilde{x}) = \sigma\int_t^T e^{-\kappa(s-t)}ds \tag{173}$$

The two processes we are interested in, can now be expressed as:

$$d\tilde{x}_t = [-\kappa\tilde{x}_s - \sigma^2\int_t^T e^{-\kappa(s-t)}ds]dt + \sigma dW_t \quad for\ \tilde{x}_0 = 0$$

$$and \qquad (174)$$

$$d\Theta_t = \Theta_t\left(\int_t^T e^{-\kappa(s-t)}ds\right)dW_t \quad for\ \Theta_0 = 1$$

From 22 we have that $P_b(t,T) = E\left[e^{-\int_t^T x_s ds}\right] = E\left[\Theta_T e^{-\int_t^T \tilde{x}_s ds}\right]$, fo

It should be stressed that $P_b(t,T)$ is not the price we are interested in, instead I will refer to $P_b(t,T)$ as our basic price - just as x is our basic interest rate process. The "true" price is derived by appropriate adjustment of the basic process x for each T in order for $P_b(t,T)$ to be equal to the market price P(t,T).

It now follows that using Girsanovs transformation in the simulation procedure requires the simultanous simulation of the two processes in formula 27.

In the main text - section 8 - we showed what the different discretizations schemed looked like for the SDE for x. Deriving the discretizations schemes for the process $\tilde{x}$ is straightforward and is for that reason left for the reader.

Deriving the actual form for the correction process Θ_t for the Euler scheme is straightforward, for that reason I will just show the order 1.5 strong Taylor discretization scheme as the others can be derived from here, ie:

$$\Theta_{k+1} = \Theta_k + \Theta_k\sigma_P(t,T)\Delta W + \frac{1}{2}\Theta_k\sigma_P^2(t,T)((\Delta W)^2 - \delta) + \frac{1}{2}\Theta_k\sigma_P^3(t,T)[\Theta_k - 1]\left(\frac{1}{3}(\Delta W)^2 - \delta\right)\Delta W$$

$$where \qquad (175)$$

$$\sigma_P(t,T) = \int_t^T e^{-\kappa(s-t)}ds$$

It follows now that the order 1 strong Milstein will be equal to the order 2 weak Milstein, and furthermore is given by the first line in equation 28[68].

Appendix H

We have that both the Extended Vasicek model and the Quadratic interest rate model is analytically very tractable.

The (semi) analytical expression for the Quadratic interest rate model is given in proposition no. 1 in the main text.

With respect to the Extended Vasicek model, can we from equation 13 in Appendix E derive the analytical solution for the price of a zero-coupon bond, ie.

$$P(t,T) = e^{B(t,T) - A(t,T)r}$$

$$for$$

$$A(t,T) = \frac{1 - e^{-\kappa(T - t)}}{\kappa} \qquad \textbf{(176)}$$

$$and$$

$$B(t,T) = \ln\frac{P(0,T)}{P(0,t)} - A(t,T)\frac{\partial \ln P(0,t)}{\partial t} - \frac{\sigma^2}{4\kappa^3}[e^{-\kappa T} - e^{-\kappa t}]^2[e^{2\kappa t} - 1]$$

Which is identical to the analytical expression for the Hull and White model, see Hull (1993 page 404).

[68] Symbolizing v by $\sigma_P(t,T)$ is not a coincidence, as the procedure here is based on a reverse application of Girsanovs transformation for a shift of probability measure. It is namely well known from the theory of contingent claim pricing that arbitrage-free pricing under the T-adjusted probability measure - for the Ornstein-Uhlenbeck process - results in a likelihood-ratio process for $\sigma_P(t,T)$. $\sigma_P(t,T)$ can because of that be recognized as the bond-price volatility in the Vasicek model and in the Extended Vasicek model - because the bond-price volatility is independent of the long-term mean value (θ or $\theta(t)$).

Using the analytical properties for these two models when pricing derivative securities is clearly efficient - both in terms of calculation speed and convergence rate. The reason for this can be explained as follows:

Let us now assume we for example wish to calculate the price of a 3-year Bermudan[69] call-option on a 10-year zero-coupon bond.

If no analytical expression is available for the spot-rate process[70] we choose as our basic pricing model - then this means we have to simulate[71] the spot-rate process all the way out to 10 years. If on the other hand an analytical expression is available for the bond-price then we only have to simulate the spot-rate process out to the time of expiry of the option - in this case 3-years.

In the analytical expression for the bond price in formula 1, the variable r is the instantaneous spot-rate. This is of course also true for the Quadratic interest rate model - namely that the spot-rate in the analytical expression is the instantaneous spot-rate.

However; in the Hull and White initial yield-curve matching-procedure the interest rates are Δt-period rates - that is equal to δ in the examples in section 9.

This means that we cannot just simulate the spot-rate process out to - for example - 3 years and then directly use the vector of simulated spot-rates and pluck them into the analytical expressions.

First we have to transform the Δt-period spot-rates into instantaneous spot-rates.

Let now r_Δ be the Δt-period spot-rate at time T^F and r be the instantaneous spot-rate at time T^F. Using equation 1 we have the following relationship in the case of the Extended Vasicek model:

$$e^{-r_\Delta \Delta t} = e^{B(T^F,T^F+\Delta t) - A(T^F,T^F+\Delta t)r} \qquad \textbf{(177)}$$

[69] The holder of a Bermudan option, also known as a limited exercise, mid-Atlantic or semi-American option, has the right to exercise the option on one or more possible dates prior to its expiry.

[70] Which for example is the case for the Black and Karisinski model.

[71] Using the phrase "simulate", we are referring to either a Monte Carlo simulation procedure or a lattice-based method.

So that:

$$r = \frac{B(T^F, T^F + \Delta t) + r_\Delta \Delta t}{A(T^F, T^F + \Delta t)} \tag{178}$$

In the case of the Quadratic interest rate model, we have:

$$e^{-r_\Delta \Delta t} = e^{A(T^F, T^F + \Delta t) - B(T^F, T^F + \Delta t)q - C(T^F, T^F + \Delta t)q^2}$$
$$where$$
$$q = (\sqrt{r} - \psi(T^F)) \tag{179}$$

From this we can deduce that r can expressed as follows as a function of the Δt-period spot-rate:

$$r = \left(\max[V_1, V_2] + \psi(T^F)\right)^2$$
$$where$$
$$V_1 = \frac{-B(T^F, T^F + \Delta t) - \sqrt{d}}{2C(T^F, T^F + \Delta t)}$$
$$and \tag{180}$$
$$V_2 = \frac{-B(T^F, T^F + \Delta t) + \sqrt{d}}{2C(T^F, T^F + \Delta t)}$$
$$for$$
$$d = B(T^F, T^F + \Delta t)^2 + 4C(T^F, T^F + \Delta t)[r_\Delta \Delta t - A(T^F, T^F + \Delta t)]$$

In the calculation performed in section 9 for both the Extended Vasicek model and the Quadratic interest rate model, we have used equation 3 and 5 respectively in order to transform the Δt-period spot-rates into instantaneous spot-rates.

These derived instantaneous spot-rates are then used in connection with the analytical expression for each of the models to calculate the forward-price from the options exercise date to the bonds maturity date.

This technique was used both in connection with the lattice-based pricing approach and the constrained Monte Carlo simulation approach.

Appendix I

Table 1: Call-option prices in the Extended Vasicek Model using Constrained Crude Monte Carlo

	K=8	Std. Error	K=16	Std. Error	K=32	Std. Error	K=64	Std. Error	K=128	Std. Error
Euler Discretization scheme - order 1 weak Taylor approximation										
52,53	*13,120*	*0,008*	*12,995*	*0,006*	*12,939*	*0,007*	*12,907*	*0,007*	*12,892*	*0,007*
62,53	*4,810*	*0,025*	*4,685*	*0,025*	*4,620*	*0,023*	*4,600*	*0,024*	*4,592*	*0,030*
67,53	*1,799*	*0,051*	*1,735*	*0,042*	*1,682*	*0,038*	*1,667*	*0,036*	*1,663*	*0,046*
72,53	*0,412*	*0,032*	*0,386*	*0,026*	*0,369*	*0,028*	*0,354*	*0,030*	*0,357*	*0,033*
82,53	*0,005*	*0,004*	*0,003*	*0,003*	*0,004*	*0,003*	*0,004*	*0,003*	*0,004*	*0,004*
Milstein Discretization scheme - order 2 weak Taylor approximation										
52,53	*13,124*	*0,009*	*12,996*	*0,007*	*12,937*	*0,007*	*12,909*	*0,006*	*12,889*	*0,009*
62,53	*4,798*	*0,029*	*4,692*	*0,029*	*4,635*	*0,030*	*4,601*	*0,027*	*4,597*	*0,031*
67,53	*1,794*	*0,046*	*1,721*	*0,041*	*1,679*	*0,037*	*1,658*	*0,040*	*1,662*	*0,044*
72,53	*0,421*	*0,030*	*0,381*	*0,029*	*0,360*	*0,028*	*0,345*	*0,033*	*0,356*	*0,026*
82,53	*0,005*	*0,004*	*0,004*	*0,003*	*0,004*	*0,005*	*0,003*	*0,004*	*0,003*	*0,003*

Table 2: Call-option prices in the Quadratic Interest Rate Model using Constrained Crude Monte Carlo

	K=8	Std. Error	K=16	Std. Error	K=32	Std. Error	K=64	Std. Error	K=128	Std. Error
Euler Discretization scheme - order 1 weak Taylor approximation										
52,53	*13,104*	*0,003*	*12,988*	*0,002*	*12,932*	*0,002*	*12,905*	*0,002*	*12,890*	*0,002*
62,53	*4,537*	*0,007*	*4,422*	*0,006*	*4,365*	*0,006*	*4,338*	*0,008*	*4,324*	*0,006*
67,53	*0,986*	*0,023*	*0,921*	*0,025*	*0,875*	*0,020*	*0,865*	*0,024*	*0,860*	*0,019*
72,53	*0,024*	*0,005*	*0,018*	*0,005*	*0,016*	*0,005*	*0,016*	*0,004*	*0,016*	*0,004*
82,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
Milstein Discretization scheme - order 2 weak Taylor approximation										
52,53	*13,104*	*0,002*	*12,988*	*0,002*	*12,932*	*0,002*	*12,904*	*0,002*	*12,891*	*0,002*
62,53	*4,535*	*0,005*	*4,421*	*0,006*	*4,366*	*0,007*	*4,339*	*0,005*	*4,325*	*0,005*
67,53	*0,991*	*0,019*	*0,913*	*0,022*	*0,882*	*0,018*	*0,868*	*0,016*	*0,855*	*0,014*

72,53	*0,024*	*0,005*	*0,018*	*0,005*	*0,015*	*0,005*	*0,015*	*0,004*	*0,016*	*0,005*
82,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*

Appendix J

Table 1: Call-and Put-option prices in the Extended Vasicek Model using Constrained Brownian Bridge Monte Carlo Simulation

	K=8	Std. Error	K=16	Std. Error	K=32	Std. Error	K=64	Std. Error	K=128	Std. Error
Call-Option prices										
52,53	*13,135*	*0,008*	*13,006*	*0,010*	*12,950*	*0,007*	*12,918*	*0,009*	*12,904*	*0,006*
62,53	*4,840*	*0,027*	*4,723*	*0,034*	*4,677*	*0,028*	*4,640*	*0,031*	*4,629*	*0,031*
67,53	*1,902*	*0,042*	*1,805*	*0,047*	*1,756*	*0,049*	*1,745*	*0,052*	*1,715*	*0,047*
72,53	*0,455*	*0,034*	*0,427*	*0,038*	*0,408*	*0,035*	*0,398*	*0,035*	*0,392*	*0,024*
82,53	*0,006*	*0,004*	*0,007*	*0,006*	*0,005*	*0,003*	*0,005*	*0,004*	*0,004*	*0,004*
Put-Option prices										
52,53	*0.000*	*0.000*	*0.000*	*0.001*	*0.000*	*0.000*	*0.000*	*0.001*	*0.000*	*0.000*
62,53	*0.296*	*0.025*	*0.309*	*0.025*	*0.311*	*0.027*	*0.310*	*0.026*	*0.318*	*0.024*
67,53	*1.627*	*0.038*	*1.668*	*0.036*	*1,680*	*0.036*	*1.709*	*0.045*	*1.687*	*0.038*
72,53	*4.503*	*0.035*	*4.590*	*0.025*	*4.630*	*0.025*	*4.654*	*0.028*	*4.665*	*0.019*
82,53	*12.629*	*0.008*	*12.757*	*0.007*	*12.810*	*0.006*	*12.842*	*0.005*	*12.857*	*0.006*

Table 2: Call-and Put-option prices in the Quadratic Interest Rate Model using Constrained Brownian Bridge Monte Carlo Simulation

	K=8	Std. Error	K=16	Std. Error	K=32	Std. Error	K=64	Std. Error	K=128	Std. Error
Call-Option prices										
52,53	*13,109*	*0,002*	*12,992*	*0,003*	*12,936*	*0,002*	*12,908*	*0,003*	*12,895*	*0,002*
62,53	*4,543*	*0,006*	*4,429*	*0,007*	*4,374*	*0,008*	*4,344*	*0,006*	*4,332*	*0,007*
67,53	*1,038*	*0,020*	*0,956*	*0,022*	*0,917*	*0,025*	*0,904*	*0,026*	*0,889*	*0,024*
72,53	*0,028*	*0,005*	*0,023*	*0,006*	*0,021*	*0,006*	*0,020*	*0,005*	*0,019*	*0,005*
82,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*

Put-Option prices										
52,53	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*	*0,000*
62,53	*0,021*	*0,005*	*0,021*	*0,004*	*0,022*	*0,006*	*0,022*	*0,007*	*0,022*	*0,005*
67,53	*0,797*	*0,020*	*0,839*	*0,020*	*0,855*	*0,020*	*0,875*	*0,024*	*0,876*	*0,021*
72,53	*4,090*	*0,005*	*4,203*	*0,005*	*4,256*	*0,005*	*4,281*	*0,003*	*4,294*	*0,004*
82,53	*12,646*	*0,002*	*12,763*	*0,002*	*12,818*	*0,002*	*12,846*	*0,002*	*12,861*	*0,002*